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SPECIFICATION OF ATMOSPHERIC LASER PROPAGATION EXPERIMENTS

Volume II

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FOREWORD

This document is the concluding report of Task II, Experiment Specification, of the Laser Space Communications System (LACE) Study. It was prepared by the Space and Information Systems Division of North American Aviation, Inc. This report is submitted in accordance with requirements of Contract NASw-977, Supplemental Agreement, dated 15 February 1965.

ABSTRACT

This document comprises the conclusion of Task II, Experiment Specification, of the Laser Space Communications System (LACE) Study. As such, it is a continuation and expansion of earlier published work under this same contract.

The areas of study covered in this report include: (1) the description of a measurement approach based upon the integration of optical effects, experimental variables, and basic interpretations of beam fluctuations; (2) the establishment of experiment and measurement requirements based upon a detailed accuracy analysis of each experimental approach; and (3) the analysis of alternate approaches to the experiment design.

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INTRODUCTION

SCOPE

The overall objective of the Laser Space Communications System (LACE) Study is to provide a plan for the implementation of a comprehensive experimental program to determine atmospheric effects on laser propagation, with particular emphasis on effects related to optical space-ground communication. The present study effort is a continuation and expansion of earlier work under this contract and is divided into four tasks: Task I, Problem Definition; Task II, Experiment Specification; Task III, System Implementation Study; and Task IV, Program Specification. The results of all four tasks also will be summarized in a final report.

This report is concerned with the conclusion of earlier effort initiated under Task II. The earlier contractual effort on this task (Reference 1) concentrated on the design of experiments to satisfy requirements defined in Task I. This effort included the discussion of the significance of a number of recommended experiments, methods of measurements, pertinent test signals and analysis, and associated block diagrams for each concept.

The areas of investigation included in this report were established with the following objectives: (1) the description of a measurement approach based upon the integration of optical effects, experimental variables, and basic interpretations of beam fluctuations; (2) the establishment of experiment and measurement requirements based upon a detailed accuracy analysis of each experimental approach; and (3) the analysis of alternate approaches to the experiment design.

SUMMARY OF CONTENTS AND CONCLUSIONS

The measurement approach section includes a generalized discussion of the interrelationships between the optical effects, experimental variables and individual variables. Also included in this section is a limited discussion of some basic interpretations of beam fluctuations in a turbulent atmosphere; a glossary of terms related to this area of study; and proposed modifications in the approach to the meteorological experiments.

The Experimental Description, Requirements, and Accuracy Analysis Section contains a detailed discussion of each of the experimental approaches. Each experiment is presented as a separate package and includes: 1) the objective; 2) experimental procedure (including a block diagram); 3) a description and discussion of the experiment; 4) a summary of all the experimental errors; 5) a detailed error analysis; and 6) a measurement analysis.

The main error sources discussed are the result of the following component and implementation considerations: beam splitters; piezo electric crystals; entrance aperture; signal to noise and gain; background; measurement time and number of measurements; mechanical vibration; tracking; circuitry maladjustments; modulators; and spot size. The analysis for experiment 1 is treated in considerably more detail than the remaining experiments as it provided the basis for some of the later analysis and derivations. Since experiment 18 (Communication Link Error Rate) is mainly a system test it has not been treated further in the present Task II report, but will be covered under Task III.

MEASUREMENT APPROACH

The purpose of this section is to provide a smooth transition between the detailed theoretical work of Task I, Problem Definition, and the more applied relationships of the experiment specification. Therefore, this section contains a discussion of atmospheric optical effects and experimental variables as they relate to the recommended experiments. Also included is an interpretation of beam fluctuations and a glossary of terms, which together summarize some of the basic concepts and definitions. Lastly, a discussion is presented on the utilization of meteorological data, and justification for the modification of previously specified meteorological experiments.

OPTICAL EFFECTS AND EXPERIMENTAL VARIABLES

The principal atmospheric optical effects affecting laser space communications are scattering and absorption by atmospheric gases and suspended liquid and solid particles, and the random modulation of the amplitude, phase, and polarization of the laser beam by refractive index inhomogeneities produced by atmospheric turbulence. The chart of Figure 1 exhibits the relationship of the experiments to these effects. The rounded boxes contain general effects. The rectangular boxes contain particular effects, such as random modulation of amplitude, phase, and polarization; particular factors, such as the sun and the moon; and the combined results on systems, such as link effects. The hexagonal boxes contain the numbers of particular experiments and terms describing the distinguishing features.

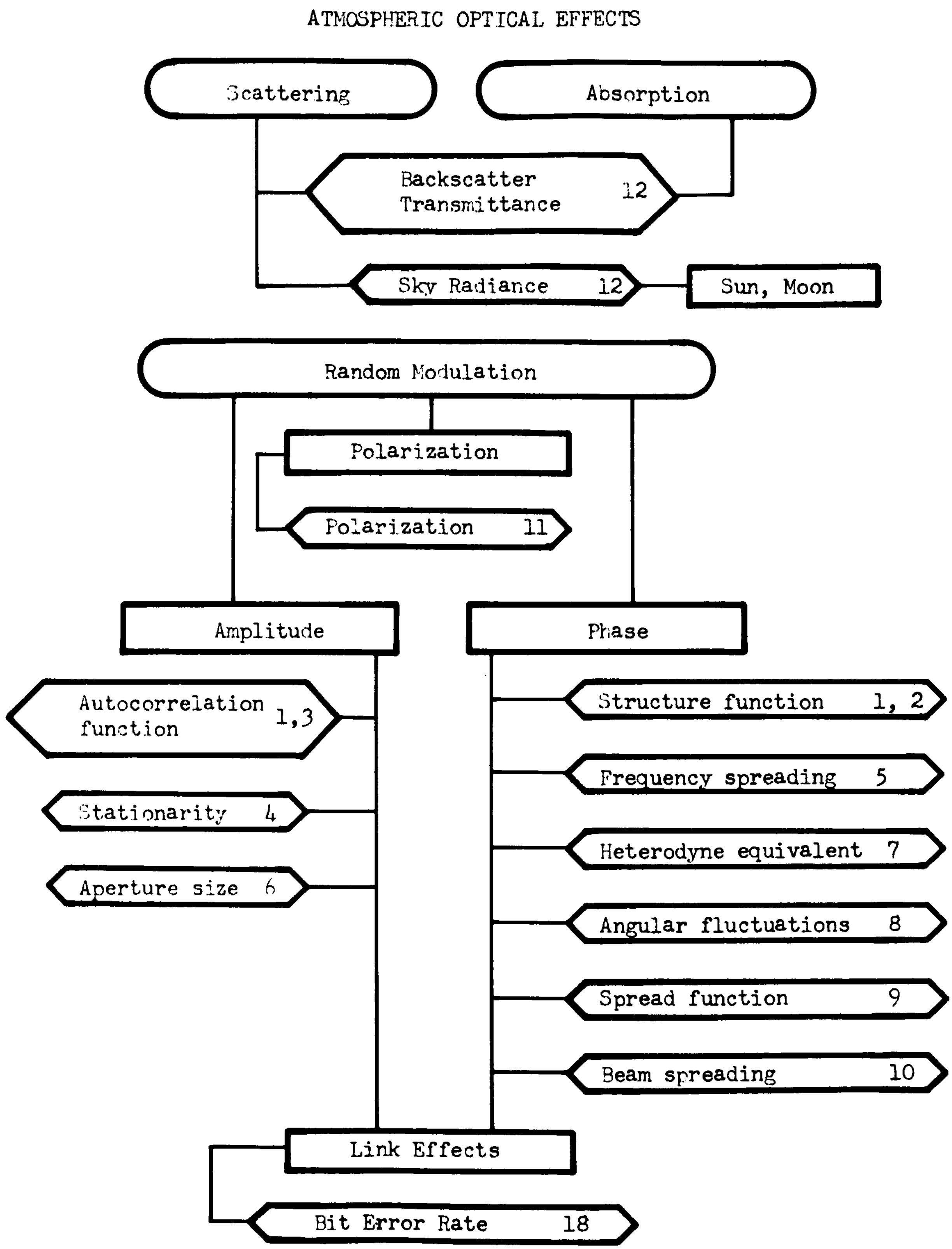

Figure 1. Relationship of Experiments to Atmospheric Optical Effects

As the figure shows, most of the experiments deal with various aspects of amplitude and phase fluctuations. One each is allocated to polarization fluctuations, communication link effects produced by amplitude and phase fluctuations, and the better known area of absorption and scattering phenomena.

The four amplitude fluctuation experiments are concerned with the statistics of the fluctuations as exemplified by the spatial autocorrelation function of the logarithm of the amplitude, with the stationarity and isotropy of the statistics, and with the modification of them by the entrance aperture of the optical system.

The seven phase fluctuation measurements look at the phase fluctuations from the point of view of the spatial structure function, frequency spreading of the laser line, heterodyne efficiency, angular fluctuations, atmospheric spread function, and beam spreading.

The communication links effect experiment, which measures bit error rates, is concerned with the systems effects of amplitude and phase fluctuations.

The scattering and absorption experiment measures backscatter, transmittance, and sky radiance. The last mentioned depends upon the relative positions of extraneous light sources, such as the sun during the day and the moon at night.

Some of the variables common to all the experiments are listed as follows.

Environment	Laser Beam	Receiver	Path
Synoptic conditions	Wavelength	Aperture size and shape	Length
Local weather	Radiant intensity	Detector time constant	Zenith angle
Terrain features	Distribution		Height
			Motion relative to air

The environment is the atmosphere, which can be described in terms of synoptic, or large-scale weather conditions, local conditions, and terrain features which affect the atmosphere. Important laser beam parameters are the wavelength and the distribution of radiant intensity as a function of angle off the beam axis. The principal characteristics of the receiver are the size and shape of the entrance aperture and the detector time constant. Significant features of the optical path are its length, zenith angle, height above the ground, and motion relative to the air. The relative motion, which affects the fluctuation temporal spectrum, depends upon wind speed and the velocities of the transmitter and receiver.

Other variables are important for particular experiments - the separation of the two apertures in the correlation and structure function measurements and the polarization in the polarization experiment. These special variables are revealed in the experiments in which they apply.

INTERPRETATION OF BEAM FLUCTUATIONS

A laser beam passing through the atmosphere is disturbed both spatially and temporally. The beam can be regarded as an electromagnetic wave, and the disturbances are interpreted in terms of wave properties.

Let us consider how the electric field of the wave looks in a plane perpendicular to the unperturbed direction of propagation, neglecting the polarization of the wave for the sake of simplicity. The electric field of the unperturbed wave would be a periodic function of time so it can be expressed in the form

$$F(t) = A \cos (2\pi\nu t + \phi),$$

where F is field strength, t is time, A is the amplitude, ν is the frequency, and ϕ is the phase angle. The atmosphere introduces spatial and temporal variations into A and φ. At a fixed point in the plane, the perturbed wave exhibits irregular temporal variations of A and ϕ at frequencies in the range of 1 to 1000 cps. At a fixed instant, A and ϕ vary in an irregular manner from point to point throughout the plane. In sum, the amplitude and phase angle have become random functions of space and time.

The properties of the random amplitude and phase angle are best described statistically. In this connection, different averaging procedures may be used. Because A and ϕ are functions of space and time, we can average with respect to space, time, or both together. In addition, we can sort values into groups, or ensembles, according to some criterion, and we can ensemble-average by taking the average of an ensemble. Theoretical discussions frequently employ an idealized form of ensemble averaging. In the following, we shall employ temporal averaging, which we shall designate by an overbar.

A fluctuation is the continually changing deviation from the average value. Thus, the phase fluctuation $\Delta\phi$ is defined by the relation

$$\phi = \bar{\phi} + \Delta\phi.$$

Likewise, the amplitude fluctuation ΔA is given by

$$A = \bar{A} + \Delta A .$$

The theoretical treatments often deal with log A, the natural logarithm of A, rather than with A itself. Then L, the log amplitude fluctuation, is specified by

$$L = \log A - \log A_0 = \log (A/A_0),$$

where A_0^2 is equal to $\langle A^2 \rangle$ and proportional to the average irradiance.

The probability distribution function is one of the properties of interest. This function describes the probability that the value of the fluctuation does not exceed a specified value. For example, if the probability distribution function P(X) of the fluctuation quantity X if normal, or gaussian, it is given by

$$P(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{X} \exp - \frac{x^2}{2\sigma^2} \, dx .$$

In the present situation, X could be $\Delta\phi$ or L.

To describe the beam fluctuations more fully, we must state the relationships between simultaneous fluctuations at two different points in the plane perpendicular to the direction of propagation. This is commonly done in terms of either correlation functions or structure functions. Let the coordinates x and y describe position in the plane, with subscripts to identify a particular point and the fluctuation quantities at it. For example, we assume that $\Delta\Phi_1$ and L_1 and $\Delta\Phi_2$ and L_2 are the fluctuations occurring at the same time at the points (x_1, y_1) and (x_2, y_2) respectively. Then the phase autocorrelation function $C_{\Phi\Phi}$ and the log amplitude autocorrelation function C_{LL} are defined by the relations

$$C_{\Phi\Phi} = \overline{\Delta\Phi_1 \ \Delta\Phi_2} \ ,$$

$$C_{LL} = \overline{L_1 \ L_2} \ .$$

In general, the autocorrelation functions in the plane are functions of the four coordinates x_1, y_1, x_2, and y_2. In some circumstances, however, average values are independent of position, and the random field in the plane is then said to be homogeneous. For example, the phase fluctuations are homogeneous in the plane if $\bar{\Phi}$, $\overline{(\Phi)^2}$, and $C_{\Phi\Phi}$ are independent of position. In this case C would reduce to a function of the coordinate differences $x_2 - x_1$ and $y_2 - y_1$. The random field in the plane is statistically isotropic if it is independent of direction as well as being homogeneous. Then the autocorrelation function becomes a function of the distance between the two points, which is given by

$$\rho = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Obviously, the isotropic field is the simplest because the autocorrelation function is a function of only one variable.

Even though the random field of the fluctuations is not homogeneous, it may approximate to homogeneity at small separations between the two points. The field is then said to be locally homogeneous, that is, homogeneous within a small region. Likewise, local isotropy may exist within a locally homogeneous region.

Closely related to the autocorrelation function is the structure function, which is defined as the mean value of the square of the difference between the fluctuating quantities at two points. In particular, the phase structure function is given by

$$D_{\phi\phi} = \overline{(\phi_1 - \phi_2)^2}$$

In an isotropic field or in a locally isotropic area, the phase structure function becomes a function of ρ which is related to the autocorrelation function as

$$D_{\phi\phi}(\rho) = 2 \left[C_{\phi\phi}(0) - C_{\phi\phi}(\rho) \right] ,$$

since

$$C_{\phi\phi}(0) = \overline{(\Delta\phi)^2}$$

Instead of considering the fluctuations at two points at the same time, we might wish to consider them at the same point at two different instants. The random field is said to be stationary if the average values are independent of time. Then, the temporal autocorrelation function is a function of the interval between the two instants. For example, the amplitude autocorrelation function is given by

$$C_{AA}(\tau) = \overline{\Delta A\ (t)\,\Delta A\ \ (t + \tau)}$$

where t and t + τ are the two instants considered. Then the temporal power spectrum $w_A(f)$ of the amplitude fluctuations is given by

$$w_A(f) = 4\int_0^\infty C_{AA}(\tau)\cos(2\pi\ f\ \tau)d\ \tau ,$$

where f is frequency, as in cycles per second. The power spectrum indicates how the fluctuations are distributed among different frequencies.

To recapitulate, the atmosphere perturbs laser beams passing through, and these perturbations can be regarded as fluctuations in the amplitude and phase of an electromagnetic wave. The statistical properties of these fluctuations represent the quantities to be derived from theory and to be determined by experiment. Among these properties are the probability distribution functions, correlation functions, structure functions, and the power spectra of the amplitude and phase fluctuations.

Other disturbances are produced in addition to amplitude and phase fluctuations. Some of these, such as the loss of coherence and directional fluctuations, might be regarded as more involved functions of the phase fluctuations. On the other hand, the possibility of polarization fluctuations would require treating the electromagnetic wave as a vector rather than a scalar.

In interpreting measurements of the beam fluctuations, we should consider that the instruments themselves perform averaging and smoothing of the fluctuations. For example, the optical receiver collects the light included within the entrance aperture and focuses it to a small image upon the detector. The result is to spatially average the fluctuations over the area of the aperture. The detector has a finite speed of response or a time constant, which causes higher frequencies to be averaged out. Consequently, aperture area and time constant must be considered as experimental parameters.

GLOSSARY OF TERMS

During the conduct of this task, it became evident that a glossary of terms and expressions associated with the study of optical propagations through a turbulent medium would be desirable. For this reason, a brief listing has been compiled of some of the terms most commonly used in this analysis. It is recognized that this listing is not complete and also contains some elementary definitions. However, it was felt that this would not detract from its overall usefulness in improving the clarity of the report; and in providing the beginning of a more comprehensive set of definitions and nomenclature on this subject.

Glossary

Amplitude - magnitude of electromagnetic wave, such as A of A cos $(\omega t + \phi)$.

Aperture - opening, such as entrance aperture of optical system.

Autocorrelation function - averaged product of two values of the same variable, such as at two different points, instants, or both.

Averaging brackets - angular brackets < > used to indicate averaging of included quantity.

Bessel function - mathematical function of which those of first kind and zero or first order are designated by J_0 and J_1 respectively.

Characteristic length r_c - particular argument of structure function $D(\rho)$ such that $D(r_c)$ corrects measured autocorrelation function for finite size of aperture.

Correlation function - interchangeable with autocorrelation function in present context.

Correlation length - separation of points at which spatial autocorrelation vanishes. An isotropic autocorrelation function approaches a maximum value $C(0)$ as ρ approaches zero, and it decreases toward zero, not necessarily monotonically, as ρ goes to infinity. We can specify a scale in terms of the value of ρ which first reduces $C(\rho)$ to zero or to some fraction of $C(0)$, such a 1/e or 1/2. For example, suppose that $C(L_c)$ equals the first zero and succeeding oscillations become successively smaller. The quantity L_c is called the correlation length or distance. The correlation is large for ρ less than L_c and small for greater than L_c, and the fluctuations become uncorrelated for ρ much greater than L_c.

Correlation time - time difference at which temporal autocorrelation vanishes. The correlation time of a stationary autocorrelation function is analogous to the correlation distance of an isotropic autocorrelation function. The maximum value is C(0), and the correlation time T_c is the value of τ which first reduces $C(\tau)$ to zero or some fraction of C(0).

Correlation variables - independent variables of correlation or structure functions, such as separation distance ρ or time difference τ.

Fluctuation - deviation of random variable from the average value.

Flux, radiant - radiant energy per unit time, synonymous with radiant power; standard symbol is P.

Frequency - rate of recurrence of periodic variable, expressed by f for cycles per unit time or ω for radians per unit time.

Frequency spreading - change in frequency Δf produced by propagation through turbulent air.

Fresnel lens - a diffracting screen which focuses a point source of light, also known as a zone plate.

Homogeneity - Property of random field such that spatial autocorrelation or structure function depends upon vector separation $\vec{\rho}$ and not upon particular points.

Intensity, radiant - radiant power per unit solid angle, standard symbol J. Intensity is sometimes used in other senses, such as for irradiance.

Irradiance - radiant power incident per unit area, standard symbol H.

Isotropy - property of homogeneous random field such that spatial autocorrelation or structure function depends on distance ρ separating points regardless of direction.

Log - abbreviation for logarithm, to the base e in present context.

Log amplitude - logarithm of amplitude A, designated by L.

Log amplitude correlation function - autocorrelation function of L, expressed by $C_{LL}(\rho)$ or $C_{LL}(\tau)$ or $C_{LL}(\rho,\tau)$.

Log amplitude structure function - structure function of L, expressed by $D_{LL}(\rho)$.

Optical transfer function - fourier transform of spread function of lens or atmosphere.

Overlap area - common area of two overlapping circles, designated by $K_o(r)$.

Phase - argument $\omega t + \phi$ of cos $(\omega t + \phi)$, often used to refer to phase angle ϕ alone.

Phase angle - angle ϕ in cos $(\omega t + \phi)$.

Phase correlation function - autocorrelation function of ϕ, expressed by $C_{\phi\phi}(\rho)$.

Phase difference - $\phi_{21} = \phi_2 - \phi_1$, where ϕ_2 and ϕ_1 are phase angles at two different points or instants.

Phase structure function - mean square value of ϕ_{21} , expressed by $D_{\phi\phi}(\rho)$.

Polarization - property indicative of direction of electric field in electromagnetic wave.

Polarization angle - the angle x indicating the rotation of the plane of polarization of the turbulent air.

Power, radiant - energy per unit time, synonymous with radiant flux, standard symbol P. The following distinctions are made in the text: P_m - measured, received power including signal power P_s from the laser and background P_b ; P_t - power sent out from laser transmitter; P_r - power received within circular pinhole of radius r.

Reflectance, radiant - ratio of "reflected" radiant power to incident radiant power, standard symbol ρ .

Separation distance - distance ρ between two points, which is the independent variable of isotropic correlation or structure functions.

Separation vector - directed distance $\vec{\rho}$ between two points, which is the independent variable of homogeneous but anisotropic correlation or structure functions.

Spatial correlation function - depends upon simultaneous values at two different points, may be homogeneous and/or isotropic. An example is $C_{LL}(\rho)$.

Spread function - description of distribution of irradiance in image of point source, designated by $S(x, y)$.

Stationarity - property of random field such that temporal autocorrelation or structure function depends on time difference τ and not upon particular times. An example is $C_{LL}(\tau)$.

Strehl definition - ratio of irradiance at center of diffraction pattern to irradiance in absence of aberrations, designated by $\mathscr{D}$.

Structure function - mean square value of difference of random variable at two points or two instants, such as phase structure functions $D_{\phi\phi}(\rho)$, and $D_{\phi\phi}(\tau)$.

Temporal correlation function - depends upon values at the same point at two different instants, such as $C_{LL}(\tau)$.

Time difference - independent variable of stationary temporal correlation or structure function, denoted by τ.

Transmittance, radiant - ratio of "transmitted" radiant power to incident radiant power, standard symbol τ .

VCO output phase - phase angle Ψ of voltage-controlled, variable-frequency oscillator.

UTILIZATION OF METEOROLOGICAL DATA

Meteorological data for synoptic and local conditions can be used to identify experimental situations for the classification of results, and to provide a basis for statistical analysis of correlations between optical effects and meteorological parameters. The meteorological data are common to the optical effects which are to be investigated and should not be considered as subjects for separate experiments.

In conjunction with any number of optical effects experiments at a given site, it is recommended that atmospheric conditions be identified by a Weather Bureau summary of synoptic conditions and by visual observations at the site. The Weather Bureau summary should identify the air masses present over the areas covered by the experiment and should describe pressure patterns, fronts, jet streams, and associated weather. In addition, at the ground terminal visual observations should be made of cloud types, heights, and cover; visibility and the appearance of the sky; weather phenomena, such as fog and precipitation; and measurements of temp., pressure, humidity, wind speed, direction, and gusts.

The principal use of the meteorological data is to identify the experimental situations. This identification provides a basis for classifying experiments made at different times and different places and for estimating how representative the experimental results are. However, the meteorological data do not determine the optical effects except in a statistical way. The second use of the meteorological data, which depends upon the accumulation of optical data for achievement, is to find statistical relationships between meteorological conditions and optical effects. For example, correlations might be found with air mass type, pressure patterns, fronts, jet streams, or strong winds.

Ordinary meteorological measurements do not really specify the optical properties of the atmosphere. For example, turbulent optical propagation is determined by refractive index fluctuations. These depend upon temperature fluctuations, which are related to wind shear and temperature lapse rate in the surface layer of the atmosphere. However, a definite relationship is not yet established between the statistics of the refractive index fluctuations and ordinary meteorological variables, and extraordinary instrumentation is required to determine refractive index fluctuations by means of temperature fluctuation measurements.

Performing extraordinary measurements to determine the optical properties of the atmosphere corresponding to optical experiments would be of scientific value in providing a rigorous test of theory, but it may be beyond the scope of the LACE program. Presumably, the designer wants to know the magnitude of various optical effects, representative values, and the range of values associated with different sites. Therefore, sufficiant and adequate meteorological data should be collected to satisfy the designers requirements.

MODIFICATION OF PROPOSED METEOROLOGICAL EXPERIMENTS

Reconsideration of the experiments previously proposed as numbers 13 to 17 inclusive (Reference 1) reveals that they are concerned with identifying atmospheric conditions whereas the other experiments deal with the optical effects of atmospheric conditions. This identification of atmospheric conditions is desirable for all the optical experiments. In recognition of this commonality, we have stopped designating meteorological measurements as separate experiments, and we have incorporated experiments 13 to 17 with the others.

Experiment number 12 on backscatter now includes the measurements of sky radiance and atmospheric transmittance suggested previously in experiment 13. Part of the reason for the transfer is that the backscatter setup requires very little adaptation to measure these additional quantities. Sky radiance is measured with the laser off. Atmospheric transmittance of a certain length of atmosphere is inferred from the laser light returned by a retro-reflector placed at that distance from the backscatter setup.

The procedures recommended in experiments 14 and 15 are to be followed in identifying the environment for experiment 18 and experiments 1 through 12. In particular, Weather Bureau summaries of synoptic meteorological conditions are to be utilized, supplemented by local observations of visibility and clouds.

The measurements of experiment 16 should be carried out in conjunction with experiment 1 or 2 and 3, and the procedures of experiment 17, suggested as a very limited alternative to experiment 16, can be dropped. The gist of experiment 16 was to determine the statistics of the refractive index inhomogeneities from measurements of atmospheric temperature fluctuations, temperature, pressure, and humidity. The refractive index statistics would provide a test of analytic models of the turbulent atmosphere and a test of the propagation theories by providing input data for the theoretical formulas to see whether they give the same optical effects as are measured.

The utilization of meteorological data and the assessment of refractive index statistics by means of temperature fluctuation measurements will be discussed more fully in the systems implementation study.

EXPERIMENT DESCRIPTION, REQUIREMENTS AND ACCURACY ANALYSIS

Each experiment that was previously specified under this task in Reference 1 , has been completely reevaluated and in some cases redefined. The main emphasis has been devoted toward a thorough analysis of the experimental errors so that firm requirements for the measurement program implementation could be determined. This has resulted in the establishment of experimental approaches, for obtaining the required measurements, which are general enough to allow some flexibility in the choice of a final design for the implementation. In many cases a number of alternate approaches are suggested.

This material in this section is organized in such a way that each experiment is an integral package. For each experiment a summary of the specifications analysis is presented first, and this is followed by a detailed discussion of the measurement and error analysis.

MEASUREMENT OF SPATIAL AND TEMPORAL LOGARITHMIC AMPLITUDE CORRELATION FUNCTION AND PHASE STRUCTURE FUNCTION - EXPERIMENT NO. 1

OBJECTIVE

To obtain the spatial and temporal statistics of amplitude and phase of coherent (laser) light after propagation through the atmosphere.

EXPERIMENTAL PROCEDURE (Figure 2)

Description

The laser beam is sampled at two points $\bar{x}$ and $\bar{x} + \bar{\rho}$ lying in its cross-section. By placing a square wave driven, mirrored piezoelectric

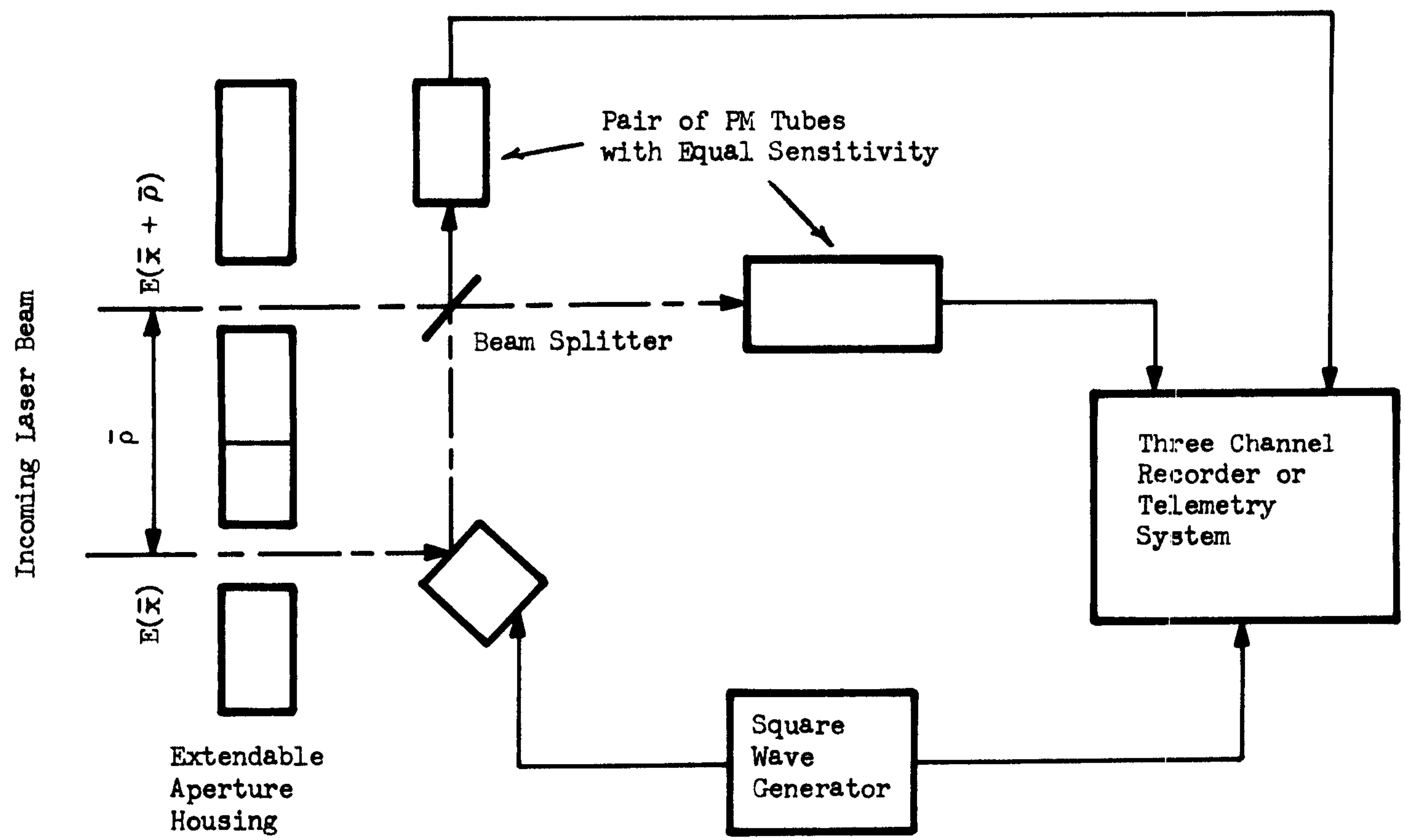

Figure 2. Measurement of Spatial and Temporal Log Amplitude Correlation Function and Phase Structure Function.

crystal in one channel and using a beam splitter in both, the two waves can be mixed on the surfaces of two photomultiplier tubes. Due to the crystals motion, the outputs of these tubes will alternately have 0, $\pm$ 90 and 180 degree phase differences. The signals so derived are used to calculate the phase difference and the individual amplitudes of the field at the sampling holes. From this data the desired statistics may be calculated.

Discussion

The information that could be gained from this experiment is substantial, but its implementation would be quite difficult. Knowledge of the phase structure function $D_{\phi\phi}(\bar{\rho})$ will be beneficial in proving or disproving various theoretical works as well as providing an insight into design problems involving heterodyne systems. The difficulty in implementation is such that very poor results would be obtained if the phase difference is greatly affected by environmental and instrumentation variations. This means that mechanically isolated systems must be used as well as highly accurate tracking and receiver stabilization techniques in the case of transmitter-receiver relative motion. From the measured amplitudes, the statistics of amplitude fluctuations can be established and used in developing incoherent systems. Here care must be exercised in receiver-transmitter alinement, as will be discussed in the section on measurement analysis.

The data output of the experiment are sets of coefficients $\{C_i\}$. For each sampling period (set member) there are four coefficients C_1, ... C_4. These values result from the four different combinations the light field at the photo detector has, due to the four different phase shifts. As will be explained in the measurements analysis section, the coefficients may be processed to give the phase structure function and log amplitude correlation function. The processing may be accomplished easily with a digital computer once the sets $\{C_i\}$ are known from recorded values.

The experiment has been devised so that both spatial and temporal statistics may be computed.

SUMMARY OF ERRORS

Phase Structure Function

Errors in measuring the phase structure function stem from many factors. We list the main ones here.

Receiver Aperture Size - Because the sampling holes have a finite size, there will be an uncertainty $2\ D_{\phi\phi}(r_c)$ in the measurement of the phase structure function. The quantity r_c is a characteristic dimension somewhat smaller than the diameter of the sampling hole.

Beam Splitter Inaccuracies - When the beam splitter does not divide power in a 50/50 ratio but in the ratio $\frac{(1-\alpha)}{(1+\alpha)}\ 0 \leq \alpha \leq 1$ the phase structure function will be in error by the factor

$$\sigma_b^2 = \frac{\alpha^2}{2\cos^2\beta}\ \left(C_{LL}(0) - C_{LL}(\bar{\rho})\right)$$

where β is the difference in the phase shift of the light passing through the beam splitter and $\pi/2$. $C_{LL}(\bar{\rho})$ is the correlation function of log amplitude. (See Appendix A for detailed analysis.)

Differences in Channel Gain - When there is a constant gain difference between the two signal processing channels of absolute gain g_1 and g_2 there will be an error in determining the phase structure function by the amount

$$\sigma_g^2 = \left(\frac{\Delta g}{2g}\right)^2 \left(C_{LL}(0) - C_{LL}(\bar{\rho})\right)$$

where $g \simeq \frac{g_1 + g_2}{2}$ and $C_{LL}(\bar{\rho})$ is the log amplitude correlation function, and $\Delta g = g_1 - g_2$

Detector Noise - When signal power and background power are present, the phase structure function will be in error by the amount

$$\sigma_n^2 = \frac{(P_s + P_B)\, e}{P_s^2\, \eta\, \tau}$$

where P_s and P_B are the signal and background average power as measured at the output of one photomultiplier, e is the electronic charge, η is tubes' quantum efficiency, τ is the integration time of the output filter.

Inaccuracies in the Piezoelectric Phase Shift - If the phase shift produced by the piezoelectric crystal is not exactly 90° and 0° then the phase structure function will be in error by

$$\sigma_p^2 = \frac{\overline{\epsilon_c^2}}{2\,(2)^{\frac{1}{2}}} + \frac{3}{4}\,\overline{\epsilon_r^2}$$

where ϵ_c is a constant error in the phase shift produced by the piezocrystal and ϵ_r is the random phase shift error produced by the piezocrystal.

Angle Tracking Error - When the receiver must be tracked by the transmitter an error will be introduced into measurement of the phase structure function by the amount

$$\sigma_T{}^2 = \left(\frac{2\pi}{\lambda}\rho\right)^2 \left\langle \gamma^2 \right\rangle_T$$

Here λ is the wavelength

ρ is the sampling hole separation

$\left\langle \gamma^2 \right\rangle_T$ is the variance of the angular error in tracking as measured over the total measurement time T.

Number of Measurements - The error in measuring the phase structure function caused by the number measurements taken, n (the set of numbers $\left\{ C_i \right\}$) decreases with increasing n as it does with the other random error terms. For example, if $D^m_{\phi\phi}(\bar{\rho})$ is the measured value of the correlation function, it is related to $D_{\phi\phi}(\bar{\rho})$ by

$$D_{\phi\phi}(\bar{\rho}) = D^M_{\phi\phi}(\bar{\rho}) \pm \left[\frac{2D^2_{\phi\phi}(\bar{\rho})}{n-2} \right]^{\frac{1}{2}}$$

as will be shown in Appendix B.

Combining all the errors, the phase structure function will equal

$$D\ (\bar{\rho}) = D^{M}_{\phi\phi}(\bar{\rho}) - 2\, D_{\phi\phi}(r_c) - \left(\frac{2\pi}{\lambda}\rho\right)^2 \left\langle \gamma^2 \right\rangle_T + \frac{\epsilon_c^{\,2}}{2\,(2)^{\frac{1}{2}}} - \frac{3}{4}\left\langle \epsilon_r^2 \right\rangle$$

$$- \left[\frac{\alpha^2}{2\cos^2\beta} + \left(\frac{\Delta g}{2g}\right)^2 \right] \left(C_{LL}\ (o) - C_{LL}\ (\bar{\rho}) \right)$$

$$\pm \left[\frac{2D^2_{\phi\phi}(\bar{\rho}) + \dfrac{(P_s + P_b)\ e}{P_s^{\,2}\ \eta\ \tau} + \frac{3}{4}\left\langle \epsilon_r^{\,2} \right\rangle}{n - 2} \right]^{\frac{1}{2}}$$

Log Amplitude Correlation Function

The errors in measuring the log amplitude correlation functions are

Receiver Aperture Size - A finite size aperture will cause an error $2D_{LL}\ (r'_c)$ in the measurement of the log amplitude correlation function. $D_{LL}\ (\bar{\rho})$ is the structure function for log amplitude and r'_c is a characteristic dimension somewhat smaller than the diameter of the sampling hole. It is not necessarily the same as r_c in the measurement of the phase structure function.

Beam Splitter Inaccuracies - When the beam splitter is inaccurate in both its power division and phase shift so that $\frac{(1 - \alpha)}{(1 + \alpha)}$, $0 \le \alpha \le 1$, is the ratio of power division and β is the difference in phase shift of the light passing through the beam and $\pi/2$, then the average error in $C_{LL}(\rho)$ will be

$$\left\langle \Delta L_b \right\rangle = \frac{-\sin\beta}{4}$$

and the variance will be

$$\sigma_B^2 = \frac{\alpha^2}{2}\left(C_{LL}(0) - C_{LL}(\bar{\rho})\right)$$

Differences in Channel Gain - If there is a difference again in channel (voltage) gains, then the average error in $C_{LL}(\bar{\rho})$ will be

$$\left\langle \Delta L_g \right\rangle = +\frac{\Delta g}{4}$$

Detector and Background Noise - Measurement of the log amplitude function will depend on average background power as well as detector noise caused by the present signal and background power. The average error in the log amplitude function will be

$$\left\langle \Delta L_{N_b} \right\rangle = \frac{P_b}{4\ P_s},$$

where P_b is the average background power and P_s is the average signal power

$$P_s = K\langle A_1^2\rangle = K\langle A_2^2\rangle,$$

K a power conversion proportionality constant. The variance in the measurement will be

$$\sigma_n^2 = \frac{3}{8 P_s} \left[\langle n_s^2 + n_b^2 \rangle \right]$$

with $\langle n_s^2 \rangle$ and $\langle n_b^2 \rangle$ the average noise power due to signal and background power.

Inaccuracies in the Piezoelectric Phase Shift - When the difference between phase shifts in the 0 and 90 degree positions of the piezoelectric is composed of a constant term and a random term then the log amplitude correlation function will have an average error

$$\left\langle \Delta L_p \right\rangle = \frac{\epsilon_{pc}}{4\ (2)^{1/2}}$$

and a variance

$$\sigma_p^2 = \frac{\langle \epsilon_{pr}^2 \rangle}{32}$$

Combining all errors, the log amplitude correlation function will be

$$C_{LL}(\bar{\rho}) = C_{LL}^{M}(\bar{\rho}) + 2\, D_{LL}(r_c) - \frac{\sin\beta}{4} - \frac{\Delta g}{4} + \frac{\epsilon_{pc}}{4\ (2)^{1/2}}$$

$$\pm \left[\frac{C_{LL}^2(0) + C_{LL}^2(\bar{\rho}) + \dfrac{3\left(\langle n_s^2\rangle + \langle n_b^2\rangle\right)}{8P_s} + \dfrac{\alpha^2}{2}\left(C_{LL}(0) - C_{LL}(\bar{\rho})\right) + \dfrac{\langle \epsilon_{pr}^2\rangle}{32}}{n - 2} \right]^{\frac{1}{2}}$$

Measurement Analysis

Let the fields at the points $\bar{x}$ and $\bar{x} + \bar{\rho}$ be given by $A_1\sqrt{2}\,\sin(\omega t + \phi_1)$ and $A_2\sqrt{2}\,\sin(\omega t + \phi_2)$. Then the four combinations of the light field at the surface of the two photomultipliers can be written as

$$C_1 = \left\langle \left(A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2)\right)^2 \right\rangle_T$$

$$C_2 = \left\langle \left(A_1 \sin(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)\right)^2 \right\rangle_T$$

$$C_3 = \left\langle \left(A_1 \cos(\omega t + \phi_1) - A_2 \cos(\omega t + \phi_2)\right)^2 \right\rangle_T$$

$$C_4 = \left\langle \left(A_1 \cos(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2)\right)^2 \right\rangle_T$$

The braces $\langle\ \rangle_T$ indicate a time average over the sampling period. The sampling rate is chosen so that the variations in $A_{1,2}$ and $\phi_{1,2}$ over this period are negligible; taking this average we have

$$C_1 = \tfrac{1}{2} A_1^2 + \tfrac{1}{2} A_2^2 + A_1 A_2 \cos(\phi_1 - \phi_2)$$

$$C_2 = \tfrac{1}{2} A_1^2 + \tfrac{1}{2} A_2^2 + A_1 A_2 \sin(\phi_1 - \phi_2) \qquad (1.1)$$

$$C_3 = \tfrac{1}{2} A_1^2 + \tfrac{1}{2} A_2^2 - A_1 A_2 \cos(\phi_1 - \phi_2)$$

$$C_4 = \tfrac{1}{2} A_1^2 + \tfrac{1}{2} A_2^2 - A_1 A_2 \sin(\phi_1 - \phi_2)$$

These coefficients correspond to average light intensities.

From these expressions we obtain

$$\frac{C_2 - C_4}{C_1 - C_3} = \frac{\sin(\phi_1 - \phi_2)}{\cos(\phi_1 - \phi_2)} \qquad (1.2)$$

$$\phi_{12} \equiv \phi_1 - \phi_2 = \tan^{-1}\left(\frac{C_2 - C_4}{C_1 - C_3}\right)$$

and

$$(C_1 - C_3)^2 + (C_2 - C_4)^2 = 4 A_1^2 A_2^2$$

Defining

$$2 C \equiv C_1 + C_2 + C_3 + C_4 = 2(A_1^2 + A_2^2) \qquad (1.3)$$

and (1.4)

$$4 B = 4 A_1^2 A_2^2$$

then (1.5)

$$A_{1,2}^2 = \frac{C}{2} \pm \left[\frac{C^2}{4} - B\right]^{\frac{1}{2}}$$

From these quantities the spatial phase structure function $D_{\phi\phi}(\bar{x}, \bar{x} + \bar{\rho}) = \left\langle \left(\phi(\bar{x}) - (\bar{x} + \bar{\rho})\right)^2 \right\rangle$ and the spatial log amplitude correlation function $C_{LL}(\bar{x}, \bar{x} + \bar{\rho}) = \left\langle \log \frac{A_1(\bar{x})}{A_o} \log \frac{A_2(\bar{x} + \bar{\rho})}{A_o} \right\rangle$ are evaluated.

It is only possible to determine the temporal log amplitude correlation function from the recorded values when the sampling sequence is fast compared to the rate of change of the amplitudes. For each set of $\{C_i\}$ values we see from (equations 1.3, 1.4, and 1.5) that two amplitudes are determined. At the outset it is not known which amplitude is which. However, when the correlation of amplitudes from measurement to measurement is high, the relationship can be established. It would otherwise be necessary to rearrange the experiment in the following way. With one entrance pupil closed the amplitude fluctuations at one location are determined as a function of time. The temporal log amplitude correlation function $C_{LL}(\tau) = \left\langle \log \frac{A_1(\tau)}{A_o} \log \frac{A_1(0)}{A_o} \right\rangle$ can then be evaluated. The temporal phase structure function

$$D_{\phi\phi}(\bar{\rho},\tau) = \left\langle \left(\phi_1(\bar{x}, 0) - \phi_2(\bar{x}+\bar{\rho}, 0)\right) \left(\left(\phi_1(\bar{x},\tau) - \phi_2(\bar{x}+\bar{\rho},\tau)\right)\right) \right\rangle$$

can also be evaluated, as long as it is possible to trace the phase during evaluation.

Before we go into a detailed analysis of errors, some general consideration can be made. Equation 1.2 shows that the calculation of the phase difference does not depend on the light intensity at the two entrance pupils, therefore it is not required that $\langle A_1^2 \rangle$ be equal to $\langle A_2^2 \rangle$. This is not true as far as evaluation of the spatial (or time) log amplitude correlation function

$$C_{LL}(\bar{x}, \bar{x}+\bar{\rho}) = \left\langle \log \frac{A_1(\bar{x})}{A_o} \log \frac{A_2(\bar{x}+\bar{\rho})}{A_o} \right\rangle$$

and some thought must be given to measurements carried out under these conditions. Suppose that because of the limited extent of the laser beam, the average power density at two points was proportional to $\langle A_1^2 \rangle = \Psi^2 A_2^2$ where Ψ is a function of the coordinates of the two entrance pupils with respect to beam symmetry. Then three possibilities in the measurement procedure are open to the experimenter. He can:

(i) Adjust the entrance pupil aperture so that the average power entering at both points are the same. Hence, average amplitudes are the same.

(ii) Measure as is and determine

$$A_o^2 = \tfrac{1}{2}\left\langle A_1^2 + A_2^2 \right\rangle \text{ with}$$

$$\left\langle A_1^2 \right\rangle = \left\langle \Psi^2 A_2^2 \right\rangle$$

(iii) Measure as is and in addition measure

$$\left\langle A_1^2 \right\rangle \equiv A_{o1}^2 \text{ and } \left\langle A_2^2 \right\rangle \equiv A_{o2}^2$$

or equivalently $\left\langle A_1 \right\rangle \equiv A_{o1}$ and $\left\langle A_2 \right\rangle = A_{o2}$

In case (i),

$$C_{LL}(\bar{\rho}) = \left\langle \log\left(\frac{A_1}{A_{o1}}\right) \log\left(\frac{A_2}{A_{o2}}\right) \right\rangle$$

with

$$\left\langle A_1^2 \right\rangle = \left\langle (\Psi A_2)^2 \right\rangle = A_o^2$$

so that

$$C_{LL}(\bar{\rho}) = \left\langle \log \frac{A_1}{A_o} \log \frac{A_2}{A_o} \right\rangle$$

In case (ii), $A_1^2 \equiv A_o^2 + a_o^2 + a_1^2 \; ; \; A_2^2 \equiv A_o^2 - a_o^2 + a_2^2$

Here $a_{1,2}^2$ are zero mean random variables, A_o^2 and a_o^2 are constant terms.

Then

$$\left\langle A_1^2 \right\rangle = A_o^2 + a_o^2 \text{ and } \left\langle A_2^2 \right\rangle = A_o^2 - a_o^2$$

So

$$A_o^2 = \tfrac{1}{2}\left(\left\langle A_1^2 + A_2^2 \right\rangle\right)$$

With this the case, the measured log amplitude correlation function becomes

$$C_{LL}^{M}(\bar{\rho}) = \tfrac{1}{4}\left\langle \log \frac{(1 + a_o^2 + a_1^2)}{A_o^2} \log \frac{(1 - a_o^2 - a_2^2)}{A_o^2} \right\rangle$$

This can be arranged to equal

$$\tfrac{1}{4}\left\langle \left[\log \frac{(A_o^2 + a_1^2)}{A_o^2} + \frac{a_o^2}{A_o^2 + a_1^2}\right] \left[\log \frac{(A_o^2 + a_2^2)}{A_o^2} - \frac{a_o^2}{A_o^2 + a_2^2}\right] \right\rangle$$

by factoring $\left(1 + \frac{a_o^2}{A_o^2} \pm \frac{a_{1,2}^2}{A_o^2}\right)$ and using the fact that $\log(1+\epsilon) \approx \epsilon$ for $\epsilon \ll 1$.

Expanding the term $\frac{1}{A_o^2 + a_{1,2}^2}$ and taking the ensemble average we have

$$C_{LL}^{M}(\bar{\rho}) = \tfrac{1}{4} C_{LL}(\bar{\rho}) - \frac{a_o^4}{4 A_o^4} + \frac{\left\langle a_1^2 a_2^2 \right\rangle}{4 A_o^4} \ldots$$

where $C_{LL}(\bar{\rho})$ is the log amplitude correlation function when $\left\langle A_1^2 \right\rangle = \left\langle A_2^2 \right\rangle$; the equal power density or equal average amplitude case.

In case (iii), the measured log amplitude correlation function can be evaluated directly as

$$C^{M}_{LL}(\bar{\rho}) = \left\langle \log \frac{A_1}{A_{01}} \quad \log \frac{A_2}{A_{02}} \right\rangle$$

We must, of course, know which amplitude, A_1 or A_2 was computed from equation 1.5.

In case a difficulty arises in deciding which value computed by equation 1.5 is A_1 or A_2 it would be better to make measurements under experimental condition (1). This is true even though condition (1) required extensive calibration while (3) takes less time.

It should be obvious that higher order moments and estimates of the probability density functions of phase and amplitude could also be calculated from the coefficients $\{C_i\}$.

ERROR ANALYSIS

Phase Structure Function

General Analysis

The light intensities $\{C_i\}$ of equation 1.1 incident on the PM tubes cause currents which are amplified and recorded as the coefficients $\{c_i\}$ which are represented by the following equations.

$$\begin{aligned} c_1 &= C_1 g_1 + g_1 n_{c1} \\ c_2 &= C_2 g_1 + g_1 n_{c2} \\ c_3 &= C_3 g_2 + g_2 n_{c3} \\ c_4 &= C_4 g_2 + g_2 n_{c4} \end{aligned} \qquad (1.6)$$

where g_1 and g_2 are the gains of the two channels and may include differences in recording sensitivities, $\{n_{ci}\}$ is a measure of the noise associated with the values $\{c_i\}$. From these measured quantities, $\tan(\phi_1 - \phi_2) \equiv \tan \phi_{12}$ is calculated

$$\tan \phi_{12} = \frac{c_2 - c_4}{c_1 - c_3}$$

The calculation has an error proportional to

$$(\tan \phi_{12}) = \frac{c_2 - c_4}{c_1 - c_3} \left(\frac{\Delta c_2 - \Delta c_4}{c_2 - c_4} - \frac{\Delta c_1 - \Delta c_3}{c_1 - c_3} \right) \tag{1.7}$$

with

$$\begin{aligned} \Delta c_1 &= \Delta C_1 g_1 + C_1 \Delta g_1 + g_1 n_{c1} \\ \Delta c_2 &= \Delta C_2 g_1 + C_2 \Delta g_1 + g_1 n_{c2} \\ \Delta c_3 &= \Delta C_3 g_2 + C_3 \Delta g_2 + g_2 n_{c3} \\ \Delta c_4 &= \Delta C_4 g_2 + C_4 \Delta g_2 + g_2 n_{c4} \end{aligned} \tag{1.8}$$

The quantities $\{C_i\}$ depend on the phase angles ϕ_1 and ϕ_2; ϕ_1 is influenced by the beam splitter and ϕ_2 by the beam splitter and the piezoelectric crystal. The effect of beam splitter inaccuracies on the quantities $\{C_i\}$ and the phase structure function calculation is derived in Appendix A.

Now the error in ϕ_{12} due to gain errors, phase errors and background noise will be calculated. The error in phase can be written

$$\Delta \phi_{12} = \Delta(\tan \phi_{12}) \cos^2 \phi_{12} \tag{1.9}$$

The errors $\{\Delta C_i\}$ can be related to errors in the beam splitter phase shift and piezoelectric phase shift by

$$\begin{aligned}
\Delta C_1 &= - A_1 A_2 (\epsilon_{\pi/2} - \epsilon_b) \sin \phi_{12} \\
\Delta C_2 &= A_1 A_2 (\epsilon_o - \epsilon_b) \cos \phi_{12} \\
\Delta C_3 &= A_1 A_2 (\epsilon_{\pi/2} + \epsilon_b) \sin \phi_{12} \\
\Delta C_4 &= - A_1 A_2 (\epsilon_o + \epsilon_b) \cos \phi_{12}
\end{aligned} \tag{1.10}$$

Here ϵ_b is the phase error in the phase shift normally caused by the beam splitter which is $\pi/2$ rad.; ϵ_o and $\epsilon_{\pi/2}$ are the phase errors in the 0 and $\pi/2$ phase shift normally introduced by the piezoelectric crystal. Using equations 1.7, 1.8, 1.9, and 1.10, we have

$$\Delta \phi_{12} = \sin \phi_{12} \cos \phi_{12} \left[\frac{A_1 A_2 \cos \phi_{12} \left(\epsilon_b (g_2 - g_1) + \epsilon_o (g_1 + g_2) + C_2 \Delta g_1 - C_4 \Delta g_2 + g_1 n_{c1} - g_2 n_{c4}\right)}{c_2 - c_4} \right.$$

$$\left. + \frac{A_1 A_2 \sin \phi_{12} \left(\epsilon_b (g_2 - g_1) + \epsilon_{\pi/2} (g_1 + g_2) + C_1 \Delta g_1 - C_3 \Delta g_2 + g_1 n_{c1} - g_2 n_{c3}\right)}{c_1 - c_3} \right]$$

Using the fact that $g_1 \approx g_2$ and defining the changes in gain, Δg_1 and Δg_2, with relation to an average gain change Δg by

$$2\Delta g_1 \equiv \Delta g_1 + \Delta g_2 + \Delta g \text{ and } 2\Delta g_2 \equiv \Delta g_1 + \Delta g_2 - \Delta g$$

Thus,

$$\Delta g = \Delta g_1 - \Delta g_2 ; \text{ also, } g_1 - g_2 = \Delta g$$

Using equation (1.1) to substitute for the coefficients $\{c_i\}$ we have,

$$\Delta \phi_{12} = \epsilon_o \cos^2 \phi_{12} + \epsilon_{\pi/2} \sin^2 \phi_{12} + \frac{\Delta g}{2g} \frac{A_1^2 + A_2^2}{2 A_1 A_2} (\cos \phi_{12} - \sin \phi_{12})$$

$$+ \frac{(n_{c2} - n_{c4})}{2 A_1 A_2} \cos \phi_{12} - \frac{(n_{c1} - n_{c3})}{2 A_1 A_2} \sin \phi_{12} \qquad (1.11)$$

where $$g = \frac{g_1 + g_2}{2}$$

The error caused by ϵ_b is missing because the coefficients $\{C_i\}$ have been taken as independent of α and β, i.e., there is no error in the beam splitter. If a more lengthy calculation is done including this error only second order corrections are introduced into equation 1.11. It is accurate enough to add the effects of the beam splitter to equation 1.11 to give the total error.

The error terms themselves may be constant or random in nature. Random components in the beam splitter and channel gain are not likely to be very great and will be neglected. Using the subscripts c and r for the constant and random components of the error terms we define

$$\epsilon_b \equiv \epsilon_{bc}$$

$$\epsilon_0 \equiv \frac{\epsilon_0 + \epsilon_{\pi/2}}{2} + \epsilon_c + \epsilon_r$$

$$\epsilon_{\pi/2} \equiv \frac{\epsilon_0 + \epsilon_{\pi/2}}{2} - \epsilon_c + \epsilon_r$$

$$\Delta g \equiv \Delta g_c$$

$$n_{ci} \equiv n_{si} + n_{bi} + N_{bi}$$

Here $\frac{\epsilon_0 + \epsilon_{\pi/2}}{2}$ is merely a shift of the phase reference, n_{si}, n_{bi} and N_{bi} are proportional to the detector current fluctuation due to the signal and background light and to the average current due to the background light, respectively. Using these relations and including the error due to the beam splitter equation 1.11 becomes

$$\begin{aligned}\Delta\phi_{12} = {} & \epsilon_{bc} + \frac{\epsilon_c}{2} \cos 2\,\phi_{12} + \frac{\Delta g}{2g} \, \frac{A_1^2 + A_2^2}{2\,A_1\,A_2} (\cos\phi_{12} - \sin\phi_{12}) \\ & + \epsilon_r (\sin^2\phi_{12} + \cos^2\phi_{12}) \\ & + \frac{\cos\phi_{12}}{2\,A_1\,A_2} \left[(n_{s2} + n_{b2} + N_{b2}) - (n_{s4} + n_{b4} + N_{b4}) \right] \\ & - \frac{\sin\phi_{12}}{2\,A_1\,A_2} \left[(n_{s1} + n_{b1} + N_{b1}) - (n_{s3} + n_{b3} + N_{b3}) \right] \end{aligned} \qquad (1.12)$$

Two other sources of measurement error are tracking error and the error introduced by the finite area of the aperture (Reference 2). The first error results when the plane containing the sampling apertures is not perpendicular to the tracking boresight. In this case, if γ is the difference of the tracking angle from the true angle then the error in measuring the phase difference ϕ_{12} over the time T is

$$\sigma_T^2 = \left(\frac{2\pi\rho}{\lambda}^2\right)\sigma_{\gamma_T}^2 \ .$$ It is assumed that $\langle\gamma\rangle_T = 0$.

The second source of error has not been evaluated theoretically yet, but its influence is introduced by introducing the term $2\,D_{\phi\phi}(r_c)$.

Errors induced by the thermal and mechanical environment are not included explicitly but they would cause changes in path length and thus errors in the phase difference ϕ_{12}. These errors could be lumped into the beam splitter and piezoelectric phase errors.

Errors in gain differences may also be thought to be due to recording or reading differences in any data processing device.

The phase structure function is defined as $D_{\phi\phi}(\bar{\rho}) = \left\langle(\phi_1 - \phi_2)^2\right\rangle$, where $\phi_1 - \phi_2 = \phi_{12}$ in $D_{\phi\phi}$ is related to the measured phase difference ϕ_{12}^M by $\phi_{12} = \phi_{12}^M - \left\langle\phi_{12}^M\right\rangle$, where ϕ_{12}^M is the phase difference introduced by the equipment (different lengths of the two light paths from the apertures to the interference point). Adding the tracking error $\Delta\phi_t$ and the error due to a finite aperture $\Delta\phi_a$ to the errors $\Delta\phi_{12}$, there results $\left\langle\left(\phi_{12} + \Delta\phi_{12} + \Delta\phi_a + \Delta\phi_t\right)\right\rangle$. Using the fact that $\langle\phi_{12}\rangle = \left\langle\phi_{12}^M - \left\langle\phi_{12}^M\right\rangle\right\rangle = 0$, that $\langle \Delta\phi_a \rangle = 0$, and that $\langle \Delta\phi_t \rangle = 0$ for perfect tracking, the phase structure function $D^M_{\phi\phi}(\bar{\rho})$, determined from the measured quantities is

$$D^M_{\phi\phi}(\bar{\rho}) = D_{\phi\phi}(\bar{\rho}) + \langle \Delta\phi_{12}^2 \rangle + \langle \Delta\phi_a^2 \rangle + \langle \Delta\phi_t^2 \rangle$$

Later we will add the effects of a limited number of measurements. From the previous discussion

$$\langle \Delta\phi_a^2 \rangle = 2\, D_{\phi\phi}(r_c), \quad \langle \Delta\phi_t^2 \rangle = \left(\frac{2\pi\rho}{\lambda}\right)^2 \langle \gamma^2 \rangle_T \quad ,$$

$$\langle \Delta\phi_{12}^2 \rangle = \frac{\alpha^2}{2\cos\beta}\left(C_{LL}(o) - C_{LL}(\bar{\rho})\right) + \left(\frac{\Delta g}{g}\right)^2 \left(C_{LL}(o) - C_{LL}(\bar{\rho})\right)$$

$$+ \frac{\epsilon_c^2}{2\,(2)^{\frac{1}{2}}} + \frac{3}{4}\langle \epsilon_r^2 \rangle$$

In doing this, the effects of average background power has been assumed processed out and some higher order correlation terms have been discarded. Their addition do not greatly affect the magnitude of $\langle \Delta\phi_{12}^2 \rangle$.

The effects of average background power on the determination of the $\{c_i\}$ may be eliminated by subtracting a term proportional to $\langle N_{bi} \rangle$ from the coefficients $\{c_i\}$. From now on we disregard the $\langle N_{bi} \rangle$ terms. Since the measurements will be made at a large signal to noise ratio, S/N >> 1, the noise to signal ratio as obtained from the last terms in equation 1.12 can be neglected except for the error introduced by the finite number of measurements. Using these conditions, the measured phase structure function is given by

$$D_{\phi\phi}(\bar{\rho}) = D^{M}_{\phi\phi}(\bar{\rho}) - 2\, D_{\phi\phi}(r_c) - \left(\frac{2\pi\rho}{\lambda}\right)^2 \left\langle \gamma^2 \right\rangle_{\tau} - \frac{\epsilon_c^{\,2}}{2\,(2)^{\frac{1}{2}}}$$

$$- \frac{3}{4}\left\langle \epsilon_r^{\,2} \right\rangle + \left[\frac{\alpha^2}{2\cos^2\beta} + \left(\frac{\Delta g}{g}\right)\right]\left(C_{LL}(o) - C_{LL}(\rho)\right) \pm$$

$$\left[\frac{D^2_{\phi\phi}(\bar{\rho}) + \dfrac{(P_s + P_b)\, e}{P_s^{\,2}\,\eta\,\tau} + \dfrac{3}{4}\left\langle \epsilon_r^{\,2} \right\rangle}{n - 2}\right]^{\frac{1}{2}}$$

Discussion of the Phase Structure Function Error Terms

The beam splitter will have changing properties as it ages. However, if α and β can be determined, compensating adjustments can be made. It is best to choose a beam splitter with α small. With 95% beam splitter accuracy, $\frac{(1-\alpha)}{(1+\alpha)} = .95$ and α is about .025; the r.m.s. error in determining $D_{\phi\phi}$ is about .017 rad, which is not large.

The inaccuracies of the piezoelectric in producing the path length differences corresponding to 0 and $\pi/2$ rad; and phase shift can be judged roughly by supposing that the difference between 0 and $\pi/2$ is in error by p%, then the r.m.s. error in the phase structure function would be $p/\frac{\pi}{8}$ rad. from equation 1.12. To keep this error below .01 rad., $p \leq .025$ rad.

To make sure that this error is minimized it would be necessary to calibrate the displacement of the piezoelectric so that it moves only a distance $\frac{\sqrt{2}}{8}\lambda$. This could be done by interference experiments using a laser of the same frequency. To offset the effects of aging and slow thermal or mechanically caused changes in the path length, the displacement of the piezoelectric crystal could be electrically controlled, after the interference calibration, by appropriate signals from thermocouples and vibration analysis.

The beam splitter and piezoelectric crystal do not affect the background light because of its random phase difference and its zero coherence length. The background power is taken care of in the analysis by treating its amplitude as an independent random variable.

The error caused by the gain difference can also be estimated, the term $\left\langle \frac{A_1^2 + A_2^2}{2 A_1 A_2} \right\rangle \approx 1$ so that the term $\frac{\Delta g}{g} = .01$ gives an r.m.s. phase error of .005 radians.

For the error due to tracking we have that if $\rho = .5$ meter and $\lambda = 6328$ Å and we only want an r.m.s. error of 1 radian

$$\sigma_{\gamma_\tau} = \frac{6.328 \times 10^{-5}}{100\pi}$$

$$\simeq 2 \times 10^{-7}$$ radians; a very tight tolerance on tracking error.

The uncertainty in $D_{\phi\phi}$ due to noise and to the number of measurements of ϕ_{12} available can be reduced somewhat by using a filter of time constant τ. This time constant must be much smaller than the time the piezocrystal stays in its 0 and $\pi/2$ radian positions so that fluctuations in the variation of ϕ_1 and ϕ_2 are not averaged out. When n data points are available, the error due to the limited amount of data will be reduced by $\frac{1}{(\tau n)^{\frac{1}{2}}}$. From the standpoint of data reduction it would be easier to have n small and τ as large as possible under the constraint mentioned above.

Log Amplitude Correlation Function

The log amplitude correlation function is defined by

$$\left\langle \log \frac{A_1}{A_0} \log \frac{A_2}{A_0} \right\rangle \equiv C_{TT} \ (\bar{\rho}) \tag{1.14}$$

This can be written

$$\frac{1}{4} \left\langle \log \left(\frac{A_1}{A_0}\right)^2 \ \log \left(\frac{A_2}{A_0}\right)^2 \right\rangle \equiv C_{LL} \ (\bar{\rho}) \tag{1.15}$$

Where we assume $A_0{}^2 = \langle A_1{}^2 \rangle = \langle A_2{}^2 \rangle$. The quantities $A_1{}^2$ and $A_2{}^2$ can be found using equations 1.3 to 1.5. The error in $\left(\log \frac{A_1}{A_0} \ \log \frac{A_2}{A_0}\right)$ is given by

$$\Delta L = \frac{\Delta A_1}{A_1} \log \frac{A_2}{A_0} + \frac{\Delta A_2}{A_2} \log \frac{A_1}{A_0} \tag{1.16}$$

The expression for the log amplitude correlation function is given by

$$C_{LL} \ (\bar{\rho}) = \frac{1}{4} \left\langle \log \frac{A_2^2}{A_0^2} \ \log \frac{A_2^2}{A_0^2} \right\rangle \pm \sqrt{\sum_{i=0}^{k} \frac{(\Delta L_i)^2}{k-1}} \tag{1.17}$$

$$= C_{LL}^{M} \ (\bar{\rho}) \pm (\Delta L)_{RMS}$$

Here ΔL_i is the i th error term.

To compute ΔA_1 and ΔA_2 we use equations 1.3 to 1.5 again; then

$$4 \ A_{1,2} \ \Delta A_{1,2} = \Delta C \pm \frac{C\Delta C - 2\Delta B}{\left[C^2 - 4 \ B\right]^{1/2}}$$

with

$$C = \tfrac{1}{2}\sum_{i=1}^{4} \Delta C_i$$

$$B = \tfrac{1}{2}\Big((C_1 - C_3)(\Delta C_1 - \Delta C_3) + (C_2 - C_4)(\Delta C_2 - \Delta C_4)\Big)$$

$$\left[C^2 - 4B\right]^{\frac{1}{2}} = (A_1^2 - A_2^2)$$

$$C = A_1^2 + A_2^2$$

Substituting in these values we have

$$4A_{1,2}\,\Delta A_{1,2} = \frac{\Delta C\,(A_1^2 - A_2^2) \pm (C\Delta C - 2\Delta B)}{A_1^2 - A_2^2}$$

$$= \frac{\Delta C(A_1^2 - A_2^2 \pm (A_1^2 + A_2^2) \pm 2A_1A_2\,(\cos\phi_{12}(\Delta C_1 - \Delta C_3) + \sin\phi_{12}(\Delta C_2 - \Delta C_4)}{A_1^2 - A_2^2}$$

$$4A_1\Delta A_1 = \frac{\Delta C 2A_1^2 - 2A_1A_2\Big(\cos\phi_{12}\,(\Delta C_1 - \Delta C_3) + \sin\phi_{12}\,(\Delta C_2 - \Delta C_4)\Big)}{A_1^2 - A_2^2}$$

$$4A_2\Delta A_2 = \frac{-\Delta C 2A_2^2 + 2A_1A_2\Big(\cos\phi_{12}\,(\Delta C_1 - \Delta C_3) + \sin\phi_{12}\,(\Delta C_2 - \Delta C_4)\Big)}{A_1^2 - A_2^2}$$

The last two equations give

$$\Delta A_1 = \frac{1}{2A_1\,(A_1^2 - A_2^2)} \left[\begin{array}{l} \Delta C_1(A_1^2 - A_1A_2 \cos\phi_{12}) + \Delta C_2(A_1^2 - A_1A_2 \sin\phi_{12}) \\ \Delta C_3(A_1^2 + A_1A_2 \cos\phi_{12}) + \Delta C_4(A_1^2 + A_1A_2 \sin\phi_{12}) \end{array} \right]$$

$$(1.18)$$

$$\Delta A_2 = \frac{1}{2A_2\,(A_2^2 - A_1^2)} \left[\begin{array}{l} \Delta C_1(A_2^2 - A_1A_2 \cos\phi_{12}) + \Delta C_2(A_2^2 - A_1A_2 \sin\phi_{12}) \\ \Delta C_3(A_2^2 + A_1A_2 \cos\phi_{12}) + \Delta C_4(A_2^2 + A_1A_2 \sin\phi_{12}) \end{array} \right]$$

When all the error contributions are treated at once, the calculation becomes volumneous. We therefore will treat every error separately.

The factors which contribute errors to the measurement of the log amplitude correlation function are: a difference in channel gain, errors in positioning the piezoelectric crystal, detector and background noise, beam splitter inaccuracies, limited number of data and the finite size of the sampling hole. The last error is taken care of by addition of the error term 2 D_{LL} (r_c^1) which has yet to be evaluated theoretically. The error caused by limited data is also added; it is discussed in Appendix B. Beam splitter errors are derived in Appendix A while the other errors will be evaluated here.

For the errors caused by the piezoelectric crystal we have

$$\begin{aligned}
\Delta C_1 &= - A_1 A_2 \, \epsilon_{\pi/2} \sin \phi_{12} \\
\Delta C_2 &= A_1 A_2 \, \epsilon_o \cos \phi_{12} \\
\Delta C_3 &= A_1 A_2 \, \epsilon_{\pi/2} \sin \phi_{12} \\
\Delta C_4 &= - A_1 A_2 \, \epsilon_o \cos \phi_{12} \\
\sum \Delta C_i &= 0
\end{aligned} \tag{1.19}$$

Substituting these values into equation 1.18 yields

$$\Delta A_1 = \frac{A_1 A_2^2}{A_1^2 - A_2^2} \left(\epsilon_{\pi/2} - \epsilon_o \right) \sin \phi_{12} \cos \phi_{12}$$

$$\Delta A_2 = \frac{A_1^2 A_2}{A_2^2 - A_1^2} \left(\epsilon_{\pi/2} - \epsilon_o \right) \sin \phi_{12} \cos \phi_{12}$$

For Δ Lp we obtain

$$\Delta Lp = \left(A_2^2 \log \frac{A_2}{A_o} - A_1^2 \log \frac{A_1}{A_o} \right) \frac{\left(\epsilon_{\pi/2} - \epsilon_o \right)}{A_1^2 - A_2^2} \sin \phi_{12} \cos \phi_{12} \tag{1.20}$$

The term $\epsilon_{\pi/2} - \epsilon_o$ can be considered composed of a zero mean random term ϵ_{pr} and a constant term ϵ_{pc}. Thus $\epsilon_{\pi/2} - \epsilon_o = \epsilon_{pc} + \epsilon_{pr}$. Taking the average of ΔL_p by considering the amplitudes A_1, A_2 and phase differences ϕ_{12} independent

$$\Delta Lp = \frac{\epsilon_{pc}}{4(2)^{\frac{1}{2}}} \quad \text{since} \quad \left\langle \frac{A_2^{\,2} \log \frac{A_2}{A_o} - A_1^{\,2} \log \frac{A_1}{A}}{A_1^{\,2} - A_2^{\,2}} \right\rangle \approx -\tfrac{1}{2}$$

and

$$\left\langle \sin \phi_{12} \cos \phi_{12} \right\rangle = \frac{1}{2(2)^{\frac{1}{2}}}$$

also

$$\sigma_p^2 = \frac{\left\langle \epsilon_{pr}^2 \right\rangle}{32}$$

The contribution of the difference in gain can be found from

$$\Delta c_i = C_i \Delta g_i$$

with $\Delta g_1 = \Delta g/2$

and $\Delta g_2 = - \Delta g/2$ where Δg is the difference in (voltage) gain

$\Delta g = g_1 - g_2$

Then

$$\Delta A_1 = \frac{1}{2A_1 \left(A_1^{\,2} - A_2^{\,2}\right)} \Bigg[A_1^{\,2} \left(\frac{\Delta g}{2} (C_1 + C_2 - C_3 - C_4)\right) - A_1 A_2 \cos \phi_{12} \left(\frac{\Delta g}{2} (C_3 + C_1)\right) - A_1 A_2 \sin \phi_{12} \left(\frac{\Delta g}{2} (C_4 + C_2)\right) \Bigg]$$

Substituting for the C_i 's gives

$$\Delta A_1 = \frac{1}{2A_1 \left(A_1^2 - A_2^2\right)} \left[- \Delta g \; A_1 A_2^3 \; (\cos \phi_{12} + \sin \phi_{12}) \right]$$

Likewise

$$\Delta A_2 = \frac{1}{2A_2 \left(A_2^2 - A_1^2\right)} \left[- \Delta g \; A_2 A_1 \; (\cos \phi_{12} + \sin \phi_{12}) \right]$$

For ΔLg , the value

$$\Delta Lg = \frac{\Delta g}{2} \; (\cos \phi_{12} + \sin \phi_{12}) \; \frac{\frac{A_1^3}{A_2} \log \frac{A_1}{A_o} - \frac{A_2^3}{A_1} \log \frac{A_2}{A_o}}{A_1^2 - A_2^2} \qquad (1.21)$$

is obtained. Then

$$\langle \Delta Lg \rangle = + \frac{\Delta g}{4}$$

Next, the effects of noise are examined. The Δc_i 's are given by

$\Delta c_i = g n_i$ where n_i are the noise terms.

Since both detector and background noise are present, the result is

$$\Delta c_1 = g \left(n_{s1} + n_{b1} + N_{b1}\right)$$

$$\Delta c_2 = g \left(n_{s2} + n_{b2} + N_{b2}\right)$$

$$\Delta c_3 = g \left(n_{s3} + n_{b3} + N_{b3}\right)$$

$$\Delta c_4 = g \left(n_{s4} + n_{b4} + N_{b4}\right)$$

where n_{si} and n_{bi} are the detector (photon) noise due to the signal and background respectively and N_{bi} is due to the background power. The moments of like variables are taken as equal, i.e., $\langle n_{s_j}{}^n \rangle = \langle n_{s_i}{}^n \rangle$.

Then for the ΔA_i 's

$$\Delta A_1 = \frac{1}{2 A_1 (A_1^2 - A_2^2)} \left[A_1^2 \left(\sum_{i=1}^{4} (n_{si} + n_{bi} + N_{bi}) \right) \right.$$

$$+ A_1 A_2 \Big(\cos \phi_{12} \; (n_{s3} - n_{s1} + n_{b3} - n_{b1})$$

$$\left. + \sin \phi_{12} \, (n_{s4} - n_{s2} + n_{b4} - n_{b2}) \Big) \right]$$

$$\Delta A_2 = \frac{1}{2 A_2 (A_2^2 - A_1^2)} \left[A_2^2 \left(\sum_{i=1}^{4} (n_{si} + n_{bi} + N_{bi}) \right) \right.$$

$$+ A_1 A_2 \Big(\cos \phi_{12} \; (n_{s3} - n_{s1} + n_{b3} - n_{b1})$$

$$\left. + \sin \phi_{12} \; (n_{s4} - n_{s2} + n_{b4} - n_{b2}) \Big) \right]$$

The error in the log amplitude correlation function is then,

$$\Delta L_n = \frac{\log \frac{A_2}{A_o}}{2A_1^2\,(A_1^2 - A_2^2)} \left\{ A_1^2 \sum_{i=1}^{4} (n_{si} + n_{bi} + N_{bi}) + A_1A_2 \left[\cos \phi_{12}\,(n_{s3} - n_{s1} + n_{b3} - n_{b1}) + \sin \phi_{12}\,(n_{s4} - n_{s2} + n_{b4} - n_{b2}) \right] \right\}$$
$$+ \frac{\log \frac{A_1}{A_o}}{2A_2^2\,(A_2^2 - A_1^2)} \left\{ A_2^2 \sum_{i=1}^{4} (n_{si} + n_{bi} + N_{bi}) + A_1A_2 \left[\cos \phi_{12}\,(n_{s3} - n_{s1} + n_{b3} - n_{b1}) + \sin \phi_{12}\,(n_{s4} - n_{s2} + n_{b4} - n_{b2}) \right] \right\} \quad (1.22)$$

Then $\langle \Delta L_n \rangle = -P_b$ where P_b is the average background power. It is supposed that this can be compensated for so that it can be neglected in other computations, then

$$\sigma_n^2 = \frac{3}{8P_s} \left[\langle n_s^2 \rangle + \langle n_b^2 \rangle \right]$$

P_s is the signal power, $P_s = K \langle A_1^2 \rangle = K \langle A_2^2 \rangle$, K a proportionality constant, n_s^2 and n_b^2 are the total noise powers due signal and background.

These errors have been derived for the case of equal average power; for the other cases (cases ii) and iii) of the measurement analysis) it would be very difficult to find the exact errors. The treatment here is enough to approximate the errors in any of the cases.

Combining all the error terms results in a value of $C_{LL}(\bar{\rho})$ is

$$C_{LL}(\bar{\rho}) = C^{M}_{LL}(\bar{\rho}) + 2\,D_{LL}(r_c) - \frac{\sin\beta}{4} + \frac{\Delta g}{4}$$

$$+ \frac{\epsilon_{pc}}{4(2)^{\frac{1}{2}}} \pm \left[\frac{\left(C_{LL}^2(o) + C_{LL}^2(\bar{\rho}) + 3\,\dfrac{\left(\langle n_s^2\rangle + \langle n_b\rangle^2\right)}{8}\right.}{n-2} \right. \tag{1.23}$$

$$\left. \frac{\left. + \dfrac{\alpha^2}{2}\left(C_{LL}(o) - C_{LL}(\bar{\rho})\right) + \dfrac{\epsilon^2_{p_r}}{32}\right)}{n-2} \right]$$

PHASE STRUCTURE FUNCTION MEASUREMENT - EXPERIMENT NO. 2

OBJECTIVE

To measure directly the phase difference between two points on the wavefront of a laser beam in order to determine the phase structure for the beam. This experiment affords a simpler alternative to the phase measurement of Experiment 1.

EXPERIMENTAL PROCEDURE

Description

A laser beam is sampled at two positions separated by a distance ρ as illustrated in Figure 3. The optical frequency of one sample is altered by a modulator, and the two rays are brought into interference, resulting in a beat frequency equal to the frequency of the modulator. The phase difference between the two rays at the optical frequencies is preserved in the beat frequency. The phase difference fluctuation is then measured by means of a phase detector used in conjunction with a phase-locked loop. The phase difference is recorded continuously, and by electronically squaring and smoothing (averaging) the output, the phase structure function can be measured and recorded directly.

Discussion

After averaging out terms with optical frequencies, the output of the PM tube is proportional to

$$\tfrac{1}{2}(A_1^2+A_2^2) + A_1 A_2 \cos(\omega_{if} t + \phi_{21})$$

where A_1 and A_2 are the amplitudes of the two incident beams, $\phi_{21} = \phi_2 - \phi_1$ is the phase difference between them, and ω_{if} is the (IF) beat frequency introduced

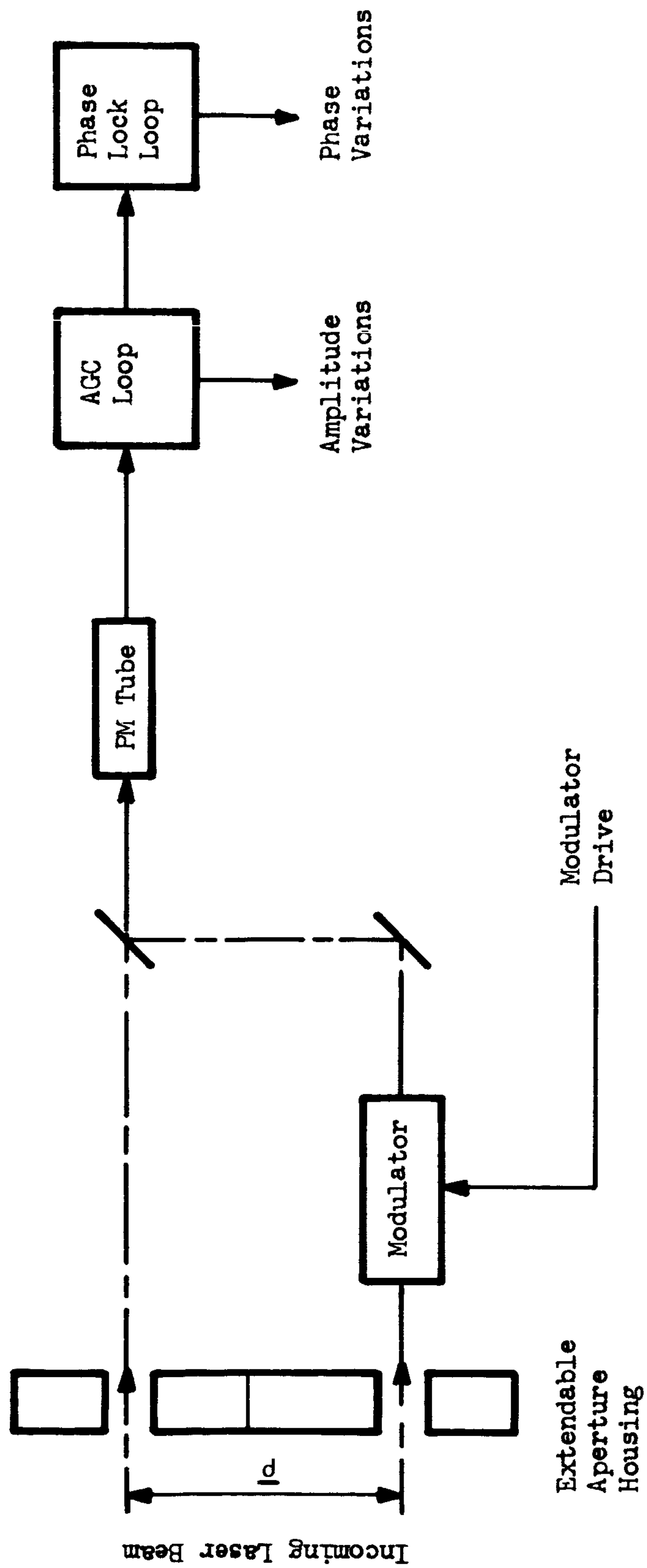

Figure 3. Measurement of Phase Structure Function

by the modulator. In order to make the output of the phase detector independent of the fluctuations in A_1 and A_2, the PM output is stabilized with an AGC amplifier which keeps the amplitude of the cosine term equal to that of the voltage controlled oscillator (VCO).

At the phase detector, the stabilized PM output is combined with the output of the VCO. The action of the phase-locked loop causes the VCO output phase Ψ to track ϕ_{21} by means of an error signal fed back from the phase detector through an amplifier and low-pass filter. The feedback loop is tight enough to keep the phase difference $\Psi - \phi_{21}$ small so the error signal is directly proportional to this difference. The output of the loop can be shown to be proportional to $d\phi_{21}/dt$, so an integrator following the loop gives an output proportional to ϕ_{21}. By squaring and averaging this output, the phase structure function $D_{\phi\phi} = \langle \phi_{21}^2 \rangle$ can thus be obtained.

SUMMARY OF ERRORS

Sources of error in this experiment include

<u>Atmospheric Doppler</u> - broadens the modulator beat frequency; the frequency spread equals approximately a few hundred cps.

<u>Receiver Aperture Size</u> - causes uncertainty of magnitude $D_{\Phi\Phi}(r_c)$, where r_c is a characteristic length somewhat smaller than the aperture diameter.

<u>Phase Tracking</u> - phase tracking loop must be capable of generating a sufficiently large tracking angle (typically on the order of 180 radians) and tracking rate (on the order of several thousand rad/sec to avoid losing lock.

Modulation Error - if the modulator frequency ω changes at the rate $\dot{\omega}$ with an average drift $\Delta\omega$, then the phase structure function will be in error by

$$\frac{1}{T}\int_0^T \left[\int_0^T (\Delta\omega + \dot{\omega} t)dt\right]^2 d\tau \qquad (2.6)$$

where T is the averaging time.

Loop Error - a random phase error ϕ_ϵ whose time rate of change is $\dot{\Phi}_\epsilon$ contributes an error to the measurement of the structure function of

$$\frac{1}{T}\int_0^T \left[\int_0^T \dot{\Phi}_\epsilon dt\right]^2 d\tau \qquad (2.7)$$

Averaging Time - the time constant T of the circuit must be large enough (on the order of a few seconds) to ensure sufficient smoothing of the output to obtain a good average.

Signal-to noise - noise from the background and from the receiver produces an error in the structure function of magnitude

$$\frac{2q\ (1+x)B}{\eta P_s} \qquad (2.9)$$

where q is the electronic charge, η is the quantum efficiency of the PM tube, B is the receiver bandwidth, and the signal power P_s and background power P_b are related by $P_b = x P_s$.

Angle Tracking Error - tracking of a moving target introduces into the structure function measurement the error

$$\frac{1}{T}\int_0^T \left[\int_0^T \frac{2\pi}{\lambda}\rho\dot{\beta}(t)dt\right]^2 d\tau \qquad (2.10)$$

Combining the errors above gives for the phase structure function:

$$D_{\Phi\Phi}(\rho) + 2D_{\Phi\Phi}(r_c) \pm \left\{ \frac{1}{T}\int_0^T \left[\int_0^T (\Delta\omega + \dot{\omega}t)dt \right]^2 d\tau + \frac{1}{T}\int_0^T \left[\int_0^T \dot{\Phi}_\epsilon dt \right]^2 d\tau \right.$$

$$\left. + \left\langle \frac{2q\ (1+x)B}{\eta P_S} \right\rangle_T^2 + \frac{1}{T}\int_0^T \left[\int_0^T \frac{2\pi}{\lambda}\rho\,\dot{\beta}\,dt \right]^2 d\tau \right\}^{\frac{1}{2}}$$

where the subscript T denotes averaging over the time T.

ERROR ANALYSIS

This section examines the effects of various sources of error on the performance of the measurement system. These sources exist both within the system itself, namely the system component, and outside of it. The latter group, which are taken up first, include the error effects arising from the random phase fluctuations due to the atmosphere. Related considerations which overlap both categories include signal-to-noise and angle tracking errors which influence the measurement of the phase structure function.

Phase Fluctuation Effects

Atmospheric Doppler

The random phase fluctuations in the laser signal, produced by atmospheric effects will appear experimentally as a frequency broadening of the beat frequency ω_{if} originating in the modulator.

The rms value of this frequency spread has been shown (Reference 4) to be of the order of a few hundred cps for typical experimental conditions. To ensure that the phase detector electronics pass a high percentage of the signal information, the design bandwidth should be several times this value, of the order of 2 or 3 kc, centered at ω_{if}. An adjustable bandwidth to allow for varying atmospheric conditions (which strongly affect the doppler shift) would be a desirable feature.

Receiver Aperture Size

The size of the two collection apertures fixes the minimum uncertainty in the measured phase difference, since the phase varies even over a small aperture. The variance of phase uncertainty is given by the phase structure function $D_{\Phi\Phi}(r_c)$, where r_c is a characteristic length which is somewhat smaller than the aperture diameter (See Glossary). $D_{\Phi\Phi}(r_c)$ depends on parameters such as path length and altitude and typically can have values of around 1 $[rad]^2$. The phase uncertainty of the two apertures contributes to the measured phase structure function $D_{\Phi\Phi}(r_c)$ the amount $2D_{\Phi\Phi}(r_c)$.

Phase Tracking

Values of the phase difference Φ_{21} between the two portions of the laser beam may range as high as a few thousand radians, and the phase tracking loop must be capable of generating a tracking angle Ψ large enough to cover this range, if the loop is not to lose lock for large phase differences. However, increasing the phase range also degrades the phase resolution (i.e. the minimum detectable phase difference), which should be approximately equal to the phase uncertainty of the wave across the collection aperture. In addition, if the ratio of phase range to phase resolution (a significant parameter) becomes too large, some serious electronic design problems may ensue. A way out of this trade-off dilemma is suggested by the intuitively evident fact that extreme values of Φ_{21} occur with very low probability, so that designing the loop to accommodate them is not a practical necessity. In support of this, it can be shown (based on the stochastic nature of the phase fluctuations) that the probability distribution P_Φ of the phase difference is gaussian. It follows that the probability that the phase exceeds $\alpha\sigma_\Phi$ is

$$P_\Phi(\alpha\sigma_\Phi) = \sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma}\int_\alpha^\infty \exp\left(-\Phi^2/2\sigma_\Phi^2\right) d\Phi \qquad (2.1)$$

where σ_ϕ is the standard deviation of Φ and α is an arbitrary factor. The rapid decrease in probability with increasing magnitude of the variable is shown by the following representative values:

$$P_\Phi(1\sigma_\phi) = 0.68$$
$$P_\Phi(2\sigma_\phi) = 0.05$$
$$P_\Phi(3\sigma_\phi) = 0.003$$
$$P_\Phi(4\sigma_\phi) = 0.00008$$

An estimate of the size of σ_ϕ has been made, which yields a value $\sigma_\phi = 180$ radians for an aperture separation of 0.5m and a 15 km path length under typical conditions.

It is of interest to inquire into the probability that the phase difference actually reaches the value $\alpha\sigma_\phi$ during the time t associated with a single measurement. To estimate the phase probability associated with a measurement time, assume the rms value of the doppler frequency spread to be 500 cps (see the above discussion on "Atmospheric Doppler"). The corresponding period, 1/500 seconds, gives a measure of the time between the random frequency shifts resulting from the phase fluctuations within the beam. More specifically, half of this period can be taken to represent the "step time" Δt of this random walk process. Thus $\Delta t = (\frac{1}{2}) \times (1/500) = (1/1000)$ sec. Now during the time t of one measurement, the phase has approximately $(t/\Delta t)$ "chances" to attain a given value of Φ. Hence, the probability $P_t(\alpha\sigma_\phi)$

of reaching the value $\alpha\sigma_\phi$ during a single measurement can be expressed by

$$P_t\ (\alpha\sigma_\phi) = 1 - \left[1 - P_\Phi\ (\alpha\sigma_\phi)\right]^{t/\Delta t} \qquad (2.2)$$

where P_Φ is the time independent phase probability function given previously. Taking $t = 1$ second, $t/\Delta t$ is 1000 and Equation 2.2 gives the values

$$P_t\ (2\sigma_\phi) = 1$$
$$P_t\ (3\sigma_\phi) = 0.95$$
$$P_t\ (4\sigma_\phi) = 0.08$$

Equation 2.2 shows that the probability of reaching a given value increases with increasing duration of the measurement, thus implying the need for an upper bound on t determined by the designed phase range of the phase-locked loop. Experimentally, the probabilities can be measured by simply recording the phase and subsequently analyzing its temporal distribution. Unless the phase tracking becomes erratic, consecutive measurements should show the same distribution.

As a final comment on this discussion of phase tracking, it is evident that the rate at which the loop can track must be much greater than the rate of change of the phase difference. A simple measure of the latter is given by the rms value of the doppler frequency spread σ_ω, which may be several thousand radians per second.

System Component Errors

In this section, the errors associated with the system components will be described.

Modulation Error

For purposes of this error analysis, the frequency ω of the frequency modulator can be written

$$\omega = \omega_{if} + \Delta\omega \tag{2.3}$$

where ω_{if} is the nominal value and $\Delta\omega$ is the average drift. As shown previously, the phase of the FM output is $(\omega t + \phi_{21})$ whose time derivative can be written, with the aid of Equation 2.3,

$$\begin{aligned} \frac{d}{dt}(\omega t + \phi_{21}) &= (\omega_{if} + \Delta\omega) + \dot{\phi}_{21} + \dot{\omega} t \\ &= \omega_{if} + \dot{\phi}_{21} + (\Delta\omega + \dot{\omega} t) \end{aligned} \tag{2.4}$$

At the phase detector, the ω_{if} term is subtracted by the action of the VCO, while the remaining terms constitute the input of the integrator, whose output is supposed to be the instantaneous value of ϕ_{21} . Thus, the frequency stability requirement for the frequency modulator is expressed by the inequality

$$|\Delta\omega + \dot{\omega} t| \ll |\dot{\phi}_{21}| \tag{2.5}$$

Since the error in the measured value of $\dot{\phi}_{21}$ is just the left side of this relation, the error in the phase structure function resulting from the modulation instability is

$$\frac{1}{T}\int_0^T \left[\int_0^\tau (\Delta\omega + \dot{\omega} t) dt\right]^2 d\tau \tag{2.6}$$

where T is the averaging time.

Photomultiplier

The signal intensity variations originating in the fluctuations of the photomultiplier gain are compensated in the AGC amplifier which follows it in the loop.

Amplifier and Filter Loop

The output V(t) of the amplifier and filter will have a random error (a constant error would not contribute to the calculation because the integration will cancel it), which causes an error Φ_ϵ in the measured phase difference:

$$\int V dt = \phi_{21} + \phi_\epsilon$$

The resulting error in the phase structure function is $\langle \phi_\epsilon^2 \rangle$

$$\frac{1}{T}\int_0^T \left[\int_0^T \dot{\phi}_\epsilon dt\right]^2 d\tau \qquad (2.7)$$

where $\dot{\phi}_\epsilon = \dfrac{d\phi_\epsilon}{dt}$

Squaring and Averaging

The integrated quantity ϕ_{21} is continuously changing. It is squared and smoothed to obtain an average. Therefore, the time constant must be larger than the longest significant periods obtained by a frequency analysis of the phase fluctuation. A time constant of a few seconds will probably be sufficient, although a variable "constant" would be preferable so it could be adjusted to the experimental conditions. By changing the setting and monitoring the output, it should be possible to find a setting where the output fluctuations are minimized.

Signal-to-Noise

Noise in the signal channel produces an error in phase tracking. At the output of the PM tube, the signal-to-noise ratio (S/N) is

$$\frac{S}{N} - \frac{\eta^2 P_s^{\,2}}{2q\eta (P_s + P_b) B} \qquad (2.8)$$

where the internal noise is assumed to be the shot noise due to the current in the PM tube and

P_s = received signal power [watts]

P_b = received background power [watts]

η = quantum efficiency [amps/watt]

q = electron charge (1.6×10^{-19} coulomb)

B = bandwidth (cps)

Letting $P_b = xP_s$, Equation 2.81 becomes

$$\frac{S}{N} = \frac{\eta P_s}{2q\,(1+x)\,B} \qquad (2.82)$$

It has been shown (Reference 5) that the variance of ϕ_{21} equals the reciprocal of the S/N ratio. Hence, from Equation 2.82:

$$\sigma_\phi = \sqrt{\left\langle \phi_{21}^{\;2} \right\rangle} = \sqrt{\frac{2q\,(1+x)\,B}{\eta P_s}} \qquad (2.9)$$

Assuming a bandwidth $B = 5000$ cps and $\eta = 2.6 \times 10^{-2}$ [amps/watt] (for an S-20 photocathode), σ_ϕ has the values for $x = 0, 1$ of

$$\sigma_\phi = 2.5 \times 10^{-7}\, P_s^{-\frac{1}{2}} \qquad (x=0)$$

and

$$\sigma_\phi = 3.5 \times 10^{-7}\, P_s^{-\frac{1}{2}} \qquad (x=1)$$

A choice of $\sigma_\phi = 1$ [radian] gives signal power requirements of $P_s = 6.2 \times 10^{-14}$ [watts] ($x = 0$) and $P_s = 12.3 \times 10^{-14}$ [watts] ($x = 1$) corresponding to an error in the phase structure function of $\sigma_\phi^{\;2} = 1\,[\text{rad}]^2$.

Angle Tracking

In order to measure the phase structure function over an up-or down-link, the measurement system must be capable of tracking the airborne or spaceborne terminal. This tracking operation introduces an error into the phase measurement.

The process of angle tracking does not involve an absolute "lock-on" but rather a continual hunting sequence in which the angular velocity of the tracking sensor alternately "overshoots" and "undershoots" the target, so that a relative angular velocity of small but varying magnitude exists at all times between the target line-of-sight and the instantaneous direction of the sensor axis. Denote this relative angular velocity by $\dot{\beta}$. The effect of $\dot{\beta}$ on the phase measurement is most easily seen by considering the corresponding lead (or lag) angle β as in Figure 4 . Since the axis of the sensor makes an angle β with the wavefront normal, it can be seen from the figure that when one portion of a wavefront arrives at a point A (representing one detector), at point B separated by a distance ρ (the second detector), the same phase front is still a distance $\rho\beta$ away, and the corresponding phase difference Φ_{AB} is

$$\Phi_{AB} = \frac{2\pi}{\lambda} (\rho\beta)$$

Hence, the phase rate error $\dot{\Phi}_\epsilon$ is given by

$$\dot{\Phi}_\epsilon = \frac{2\pi}{\lambda} \rho\dot{\beta}$$

and its contribution to the phase structure function is

$$\frac{1}{T}\int_0^T \left[\int_0^\tau \frac{2\pi}{\lambda} \rho\dot{\beta}(t)\, dt\right]^2 d\tau \qquad (2.10)$$

A representative value of $\dot{\Phi}_\epsilon$ is obtained by taking $\lambda = 6328\text{Å}$, $\rho = 0.5$ m.

$$\dot{\Phi}_\epsilon \cong \left(\frac{2\pi}{6 \times 10^{-7}}\right)(0.5)\,\dot{\beta}$$

$$\cong 5 \times 10^{6}\,\dot{\beta}\ \left[\text{rad/sec}\right].$$

If $\dot{\phi}_\epsilon$ is limited to 1 [rad/sec], the restriction on $\dot{\beta}$ becomes

$$|\dot{\beta}| < 2 \times 10^{-7} \text{ [rad/sec]}$$

or

$$|\dot{\beta}| < 0.04 \text{ [}\widehat{\text{sec}}\text{/sec]} .$$

This is a very demanding requirement.

Phase constant

Finally, it should be noted that the phase ϕ_{21} of the PM output signal includes an unknown constant which arises because the plane of the receivers does not in general coincide with the plane of the received wavefront. The angle of tilt, (which is essentially the angle β of the previous discussion) is not measured in this experiment, and so the resulting phase constant is not known. However, since it remains constant during the experiment, it is eliminated in the integration which generates ϕ_{21} and does not influence the evaluation of $D_{\phi\phi}$.

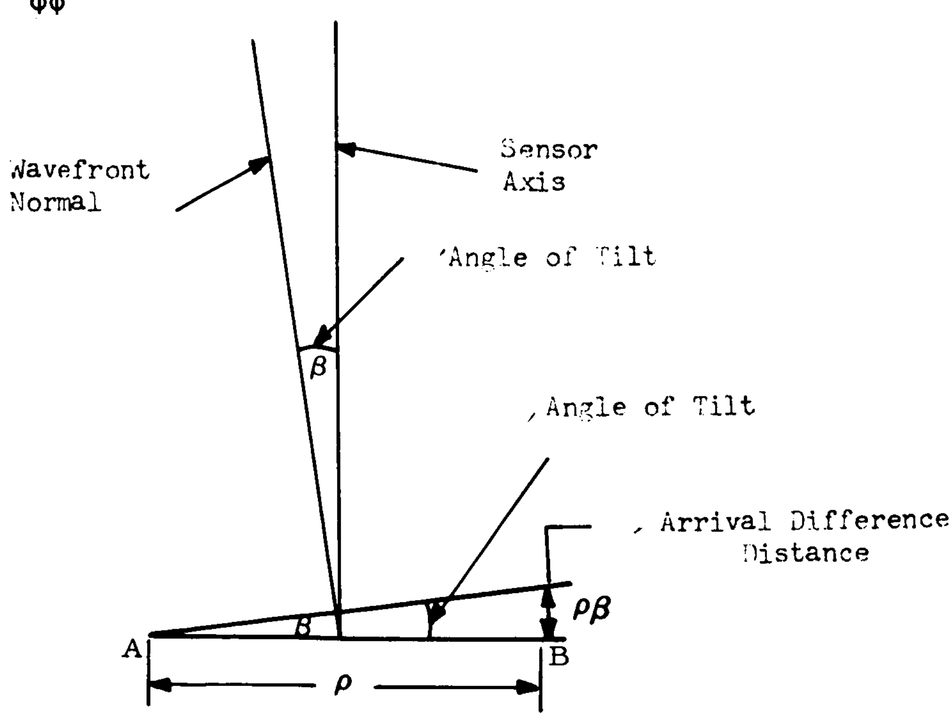

Figure 4. Wave Front and Aperture Geometry

AMPLITUDE CORRELATION MEASUREMENT - EXPERIMENT NO. 3

OBJECTIVE

To obtain the log amplitude correlation function for a laser beam after transmission through the atmosphere, from irradiance measurements made at two points within the received beam.

EXPERIMENTAL PROCEDURE

Description

The irradiance of the laser light at two points within the beam is measured by two photomultipliers (PM) which are separated by a variable distance ρ as illustrated in Figure 5. To reduce background light and secure the highest possible resolution in the measurement, narrow-band optical filters are used in front of the PM's and the receiver apertures are kept as small as the PM sensitivity permits. The outputs of the PM's, which are proportional to the square of the light amplitude at the sampled points, can be processed in any of three conceptually different ways to obtain the log amplitude correlation function $C_{LL}(\rho)$. The three methods are designated by the letters a, b, & c in Figure 5.

System a.

The outputs of the two PM-tubes are recorded for later evaluation.

System b.

The outputs of the PM-tubes are fed to logarithmic amplifiers, following which they are multiplied and averaged.

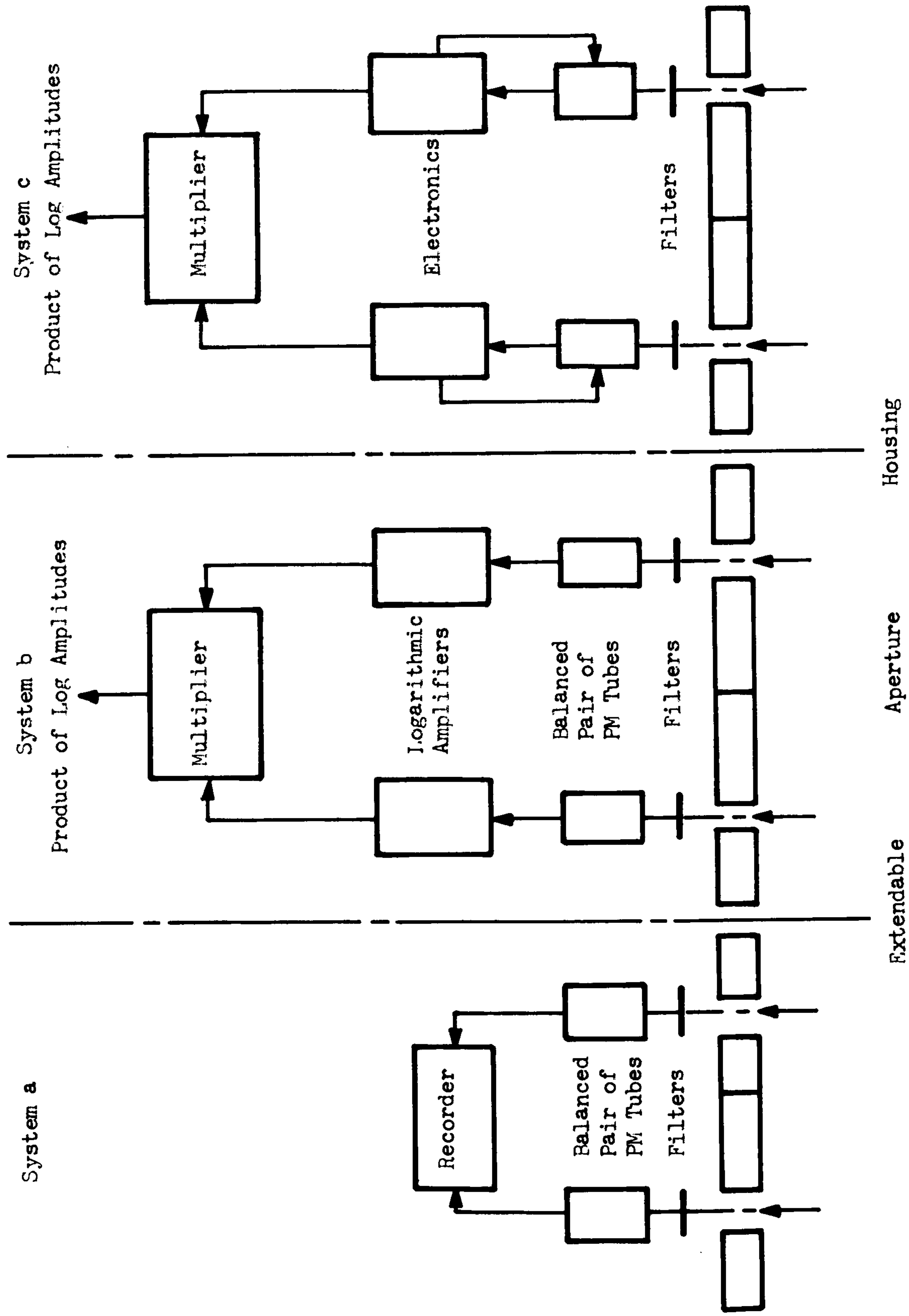

Figure 5. Measurement of Amplitude Correlation

System c.

First the average values $\langle A_1^2 \rangle$ and $\langle A_{10}^2 \rangle$ are formed. Then log A_1, log A_{10}, log A_{21} and log A_{20} are computed and the quantities (log A_1 - log A_{10}) and log A_2 - log A_{20}) are formed, multiplied together, and averaged.

Discussion

The direct recording technique (System a) is a convenient and economical procedure when computer facilities are available to process the data, and also possesses the substantial advantage of versatility. In addition to $C_{LL}(\rho)$, other functions of interest can be evaluated, including $\langle A/A_o \rangle$, $\langle \log A/A_o \rangle$ and higher moments, and the temporal correlation functions.

The other two systems require special calibration procedures which are equivalent to setting the condition log (1) = 0. In System b., multiplication with either log $\langle (A_1/A_{10})^2 \rangle$ or log $\langle (A_2/A_{20})^2 \rangle$ must give zero output, since by definition $\langle (A_1/A_{10})^2 \rangle = \langle (A_2/A_{20})^2 \rangle = 1$. An auxiliary smoothing circuit and variable gain can be used to make the calibration. This is done for each channel by increasing the time constant of the smoothing circuit until the output fluctuations are eliminated ($\langle A \rangle^2 = A_o^2$) and then adjusting the gain until the output reads zero. When the calibration has been performed on both channels, the averaged output of the multiplier is equal to $C_{LL}(\rho)$.

In System c, the calibration is accomplished by feeding to both amplifiers of Channel No. 1 the same input, A_{10}^2 and adjusting the gain until the output is zero, as required by the condition (log A_{10} - log A_{10}) = 0. Repeating with the other channel completes the calibration. The only advantage of this compared to previous systems is that changes in the average PM-tube output are automatically compensated, and no re-adjustment is required when the power level changes.

SUMMARY OF ERRORS

The following experimental factors give rise to errors in the measurement of the log amplitude correlation function:

Receiver Aperture Size - causes uncertainty of magnitude $2D_{LL}(r_c)$, where r_c is a characteristic length somewhat smaller than the aperture diameter.

Non-linearity of the PM-tubes - if the output of the PM-tube is

$$i_{pm} = a_1 P_s + a_2 \left(P_s^2 - \langle P_s \rangle^2\right),$$

where P_s is the received laser power and a_1 and a_2 are constant coefficients, then the resulting error in $C_{LL}(\rho)$ is given by

$$\pm\ 2\left(\frac{a_2}{a_1}\right)\sqrt{2\, C_{LL}(0)\left(\langle P_s^2 \rangle - \langle P_s \rangle^2\right)} \qquad (3.3)$$

Computation and Multiplication of Logarithms - if the log computation is in error by ϵ_1, ϵ_2, for channels 1 and 2, respectively, and the associated multiplication settings are only accurate to within a fraction α_1, α_2, then the output is in error by

$$\left\langle \log \frac{A}{A_o} \right\rangle \left(\alpha_1 + \alpha_2 + \epsilon_1 + \epsilon_2\right) \qquad (3.4)$$

Background Light and Internal Noise - if the background light power is P_b and the relative variance of the noise is $\frac{e}{\eta\tau\langle P_s + P_b\rangle}$, then the resulting error in the measurement of $C_{LL}(\rho)$ is

$$\pm\ 2\sqrt{C_{LL}(0)\ \frac{\langle P_s + P_b\rangle}{\langle P_s\rangle}\ \frac{e}{\eta\tau\langle P_s\rangle}} \qquad (3.6)$$

assuming $P_b \ll P_s$.

Number of Measurements - the random error terms decrease with increasing number n of repeated measurements according to the relation

$$\pm\sqrt{\frac{C_{LL}^{2}(0) + C_{LL}^{2}(\rho)}{n - 2}} \qquad (3.9)$$

Combining the errors above, the value of C_{LL} is

$$C_{LL}^{M}(\rho) + 2\ D_{LL}(r_c) + \left\langle \log \frac{A}{A_o} \right\rangle\ (\alpha_1 + \alpha_2 + \epsilon_1 + \epsilon_2)$$

$$\pm\sqrt{\frac{8\left(\frac{a_2}{a_1}\right)^2 C_{LL}(0)\left(\langle P_s^2\rangle - \langle P_s\rangle^2\right) + C_{LL}^{2}(0) + C_{LL}^{2}(\rho) + 4\ \frac{C_{LL}(0)\langle P_s + P_b\rangle e}{\langle P_s\rangle^2\ \eta\tau}}{n - 2}}$$

ERROR ANALYSIS

Errors arising from the finite size of the receiver apertures, the non-linearity of the PM-tube, from the processes of forming the logarithms and multiplying them together, from background and internal noise, and the dependence of the experimental accuracy on the number of repeated measurements are discussed in the following paragraphs.

Receiver Aperture Size

As in Experiment No. 2 (Phase Structure Function Measurement), the size of the entrance apertures determines the minimum possible resolution in the measurements. The magnitude of this uncertainty in relation to the size of $C_{LL}(\rho)$ can be expressed as $2D_{LL}(r_c)$, where r_c is a characteristic length somewhat smaller than the diameter of the aperture.

Non-Linearity of PM-Tube

Experimentally, the only requirement on the PM-tubes is that they be operated within their linear range, so that their output is proportional to the input light flux (power). The effect of a slight non-linearity can be estimated by assuming that the output characteristic of the PM-tube contains a quadratic term:

$$i_{pm} = a_1 P_s + a_2 (P_s^2 - \langle P_s \rangle^2) \qquad (3.1)$$

with

$$\langle i_{pm} \rangle = a_1 \langle P_s \rangle + a_2 (\langle P_s^2 \rangle - \langle P_s \rangle^2)$$

where i_{pm} is PM output current (amps), P_s is the received light power (watts), and a_1 and a_2 are constants. Making use of Equation 3.1, noting that the quadratic term is small in comparison to the linear term, the logarithmic output can be written

$$\log\left[\frac{i_{pm}}{\langle i_{pm}\rangle}\right] = \log\left[\frac{a_1 P_s + a_2 (P_s^2 - \langle P_s\rangle^2)}{a_1 \langle P_s\rangle + a_2 (\langle P_s^2\rangle - \langle P_s\rangle^2)}\right]$$

$$= \log\left[\frac{a_1 P_s \left(1 + \frac{a_2}{a_1}\frac{P_s^2 - \langle P_s\rangle^2}{P_s}\right)}{a_1 \langle P_s\rangle \left(1 + \frac{a_2}{a_1}\frac{\langle P_s^2\rangle - \langle P_s\rangle^2}{\langle P_s\rangle}\right)}\right]$$

$$= \log\left[\left(\frac{P_s}{\langle P_s\rangle}\right)\left(1 + \frac{a_2}{a_1}\frac{P_s^2 - \langle P_s\rangle^2}{P_s}\right)\left(1 + \frac{a_2}{a_1}\frac{\langle P_s^2\rangle - \langle P_s\rangle^2}{\langle P_s\rangle}\right)^{-1}\right]$$

$$= \log\left[\left(\frac{P_s}{\langle P_s\rangle}\right)\left(1 + \frac{a_2}{a_1}\frac{P_s^2 - \langle P_s\rangle^2}{P_s}\right)\left(1 - \frac{a_2}{a_1}\frac{\langle P_s^2\rangle - \langle P_s\rangle^2}{\langle P_s\rangle}\right)\right].$$

Retaining only terms up through first order in the multiplication, and using the approximation $\log(1 + x) \approx x$, which is valid for small x, this expression can be further simplified:

$$\log\left[\frac{i_{pm}}{\langle i_{pm}\rangle}\right] = \log\left[\frac{P_s}{\langle P_s\rangle}\right] + \frac{a_2}{a_1}\left(\frac{P_s^2 - \langle P_s\rangle^2}{P_s} - \frac{\langle P_s^2\rangle - \langle P_s\rangle^2}{\langle P_s\rangle}\right)$$

$$= \log\left[\frac{P_s}{\langle P_s\rangle}\right] + \frac{a_2}{a_1}\left(P_s + \langle P_s\rangle - \frac{\langle P_s\rangle^2}{P_s} - \frac{\langle P_s^2\rangle}{\langle P_s\rangle}\right)$$

$$= \log\left(\frac{P_s}{\langle P_s\rangle}\right) + \epsilon \tag{3.2}$$

where ϵ denotes the error term resulting from the non-linear component of the PM output.

The foregoing analysis applies equally to each receiver channel. When the logarithmic outputs of the two channels are multiplied and averaged, the result is

$$\left\langle\left(\log \frac{P_{S1}}{\langle P_{S1}\rangle} + \epsilon_1\right)\left(\log \frac{P_{S2}}{\langle P_{S2}\rangle} + \epsilon_2\right)\right\rangle$$

$$= \left\langle \log\left(\frac{P_{S1}}{\langle P_{S2}\rangle}\right)\log\left(\frac{P_{S2}}{\langle P_{S2}\rangle}\right)\right\rangle + \left\langle \epsilon_1 \log\left(\frac{P_{S1}}{\langle P_{S1}\rangle}\right) + \epsilon_2 \log\left(\frac{P_{S2}}{\langle P_{S2}\rangle}\right) + \epsilon_1\epsilon_2\right\rangle .$$

The average of the terms in the second brackets is approximately zero. The error in $C_{LL}(\rho)$, then, will be due to the variance in measuring the average. Thus, the value of C_{LL} is,

$$C^M_{LL}(\rho) \pm \sqrt{\langle\epsilon_1^2\rangle C_{LL}(0) + \langle\epsilon_2^2\rangle C_{LL}(0) + \langle\epsilon_1\epsilon_2\rangle^2}$$

$$= C^M_{LL}(\rho) \pm \sqrt{8\left\langle\left(P_s - \langle P_s\rangle\right)^2\right\rangle C_{LL}(0)\,\frac{a_2^2}{a_1^2} + \left[\text{terms} \sim \left(\frac{a_2}{a_1}\right)^4\right]}$$

$$\cong C^M_{LL}(\rho) \pm 2\,\frac{a_2}{a_1}\sqrt{2C_{LL}(0)\left(\langle P_s^2\rangle - \langle P_s\rangle^2\right)}, \tag{3.3}$$

the square-root term arising from the fact that the errors ϵ_1 and ϵ_2 are random and independent of each other.

<u>Logarithm and Multiplier Errors</u>

Other errors arise from the operations of forming the logarithms and multiplying them together. If the log computation is in error by an amount ϵ, and the multiplication setting is only accurate to within a fraction α for each channel, the effect of these errors on the output is similar to that of the PM tube, analyzed above. Thus, taking $\left\langle \log\frac{A_2}{A_{20}}\right\rangle \simeq \left\langle\log\frac{A_1}{A_{10}}\right\rangle = \left\langle\log\frac{A}{A_0}\right\rangle$,

$$\left\langle \left[\log \frac{A_1}{A_{10}} (1+\alpha_1) + \epsilon_1 \right] \left[\log \frac{A_2}{A_{20}} (1+\alpha_2) + \epsilon_2 \right] \right\rangle$$

$$= \left\langle \log \frac{A_1}{A_{10}} \log \frac{A_2}{A_{20}} \right\rangle + \left\langle (\alpha_1 + \epsilon_1) \log \frac{A_2}{A_{20}} + (\alpha_2 + \epsilon_2) \log \frac{A_1}{A_{10}} \right\rangle$$

$$= C_{LL}(\rho) + \left\langle \log \frac{A}{A_0} \right\rangle (\alpha_1 + \alpha_2 + \epsilon_1 + \epsilon_2) \tag{3.4}$$

which again consists of the desired output $\left(C_{LL}(\rho)\right)$ plus an error term. The errors α and ϵ can be minimized by choosing high quality components and then exercising care in adjusting them before the experiment. Their exact value can be determined by calibration and actual measurement of the component characteristics prior to the experiment.

Background and Internal Noise

When background power P_b (not removed by filtering) is present, the output of the PM-tubes is proportional to $P_s + P_b$. Since the background power results from an extended, incoherent source (disregarding the unlikely occurrence when an individual bright object appears in the field of view), while the laser is effectively a coherent point source, the fluctuations in P_b may be assumed negligible in comparison to the fluctuations in the signal power. If a term N due to internal noise is also included, the computation of the logarithm for either channel takes the form

$$\log \left[\frac{P_s + P_b + N}{\langle P_s + P_b \rangle} \right] = \log \left[\left(\frac{P_s + P_b}{\langle P_s + P_b \rangle} \right) \left(1 + \frac{N}{P_s + P_b} \right) \right]$$

$$= \log \left(\frac{P_s + P_b}{\langle P_s + P_b \rangle} \right) + \frac{N}{P_s + P_b}$$

where the approximation $\log(1 + x) \approx x \quad (x \ll 1)$ has again been used under the assumption that $N \ll P_s$. Making the further assumption that $P_b \ll P_s$, the evaluation of the correlation function can be written

$$\left\langle \left[\log \frac{(P_s + P_b)_1}{\langle P_s + P_b \rangle} + \frac{N_1}{(P_s + P_b)_1} \right] \left[\log \frac{(P_s + P_b)_2}{\langle P_s + P_b \rangle} + \frac{N_2}{(P_s + P_b)_2} \right] \right\rangle$$

$$= C_{LL}(\rho) \left(\frac{\langle P_s \rangle}{\langle P_s + P_b \rangle} \right)^2 \pm 2 \sqrt{C_{LL}(0) \left(\frac{\langle P_s \rangle}{\langle P_s + P_b \rangle} \right)^2 \frac{e}{\eta\tau \langle P_s + P_b \rangle}} \tag{3.5}$$

where $\left\langle \left(\frac{N}{\langle P_s + P_b \rangle} \right)^2 \right\rangle = \frac{e}{\eta\tau \langle P_s + P_b \rangle}$

e = electronic charge

η = quantum efficiency

τ = time constant

is the internal noise, the subscripts 1 and 2 designate the separate receiver channels, and the average values in the last line are the averages of both channels. The correlation functions are those for the condition of no background light. To get this true value of $C_{LL}(\rho)$ explicitly, Equation 3.5 must be multiplied through by the factor $\left[(\langle P_s + P_b \rangle) / (\langle P_s \rangle) \right]^2$ to give

$$C_{LL}(\rho) \pm 2 \sqrt{C_{LL}(0) \frac{\langle P_s + P_b \rangle}{\langle P_s \rangle} \frac{e}{\eta\tau \langle P_s \rangle}} \tag{3.6}$$

<u>Number of Measurements</u>

In common with other experimental measurements, the accuracy of C_{LL} improves as the number of repeated measurements is increased. Defining the variance Σ^2 of the products of $\log(A_1/A_{10}) \log(A_2/A_{20})$ by

$$\Sigma^2 \equiv \left\langle \left[\log(A_1/A_{10}) \log(A_2/A_{20}) - \langle \log(A_1/A_{10}) \log(A_2/A_{20}) \rangle \right]^2 \right\rangle \tag{3.7}$$

It is shown in Appendix C that

$$\Sigma^2 = C_{LL}^2(0) + C_{LL}^2(\rho) \tag{3.8}$$

so that a number n of repeated measurements give for $C_{LL}(\rho)$ a value:

$$C_{LL}^M(\rho) \pm \sqrt{\frac{C_{LL}^2(0) + C_{LL}^2(\rho)}{n-2}} \tag{3.9}$$

When the averaging is done electronically, the same equation governs the accuracy of the result, except that the number n is a function of the averaging time (time constant) τ of the circuit. The simplest procedure in this case is to make the output directly visible on an oscilloscope. By increasing τ_1 (first channel) and τ_2 (second channel) until the fluctuations of $C_{LL}(\rho)$ are minimized (which can probably be achieved with a time constant of the order of a few seconds), the accuracy of the measurement as a function of the time constant can be directly observed.

STATIONARITY TEST - EXPERIMENT NO. 4

OBJECTIVE

To test the stationarity of the log amplitude correlation function $C_{LL}(\vec{\rho})$ for a laser beam after transmission through the atmosphere, and to measure the spatial wandering of the beam. A secondary objective is to check the isotropy of $C_{LL}(\vec{\rho})$.

EXPERIMENTAL PROCEDURE

Description

As indicated in Figure 6 a Fresnel lens intercepts the incoming laser beam and is imaged by means of a field lens on the photosensitive surface of a high resolution, rapid scan image tube (vidicon or orthicon). The function of the image tube is to obtain "pictures" of the instantaneous intensity distribution across the laser beam, as a function of time. These pictures are taken at frequent intervals and recorded for later evaluation.

Isotropy is checked by making measurements for different directions of the separation vector $\vec{\rho}$.

Discussion

Stationarity, in the sense of this experiment, is tested by evaluating $C_{LL}(\vec{\rho})$ from intensity measurements made at a given pair of points (separated by $(\vec{\rho})$ as a function of time, and comparing this with the value of $C_{LL}(\vec{\rho})$ determined from an ensemble average taken over the intensity pattern at a given time. If the correlation function is stationary, the results of these two determinations should be the same, to within experimental tolerances.

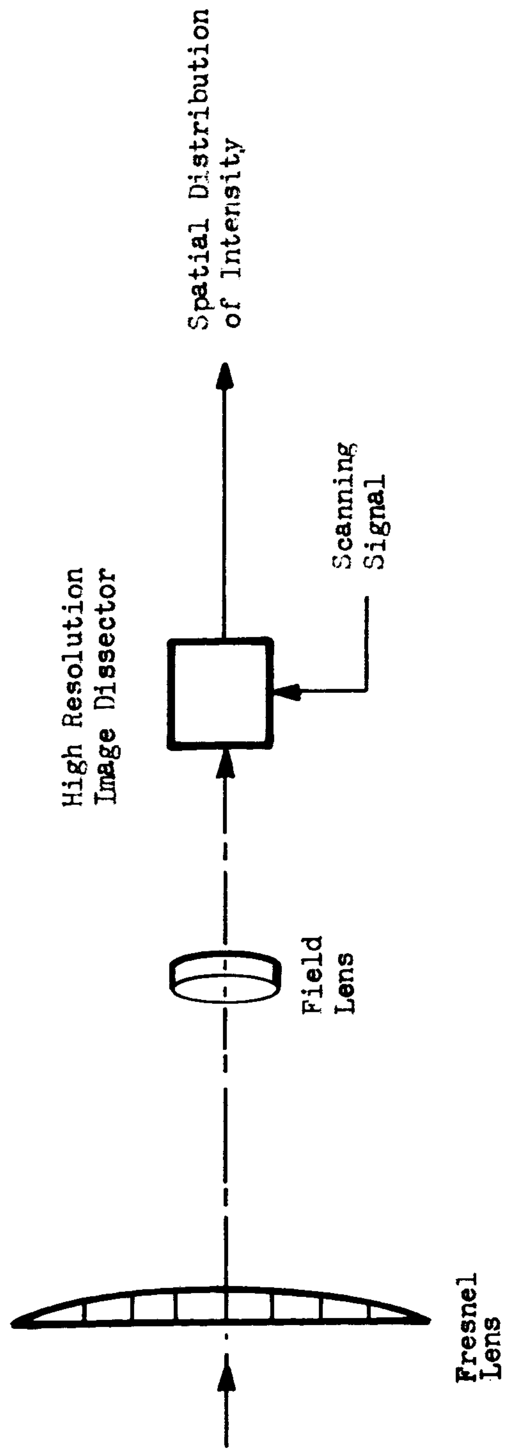

Figure 6. Stationarity Test Configuration

In making these evaluations, the orientation of $\vec{\rho}$ is kept constant, since isotropy is not assumed. The assumption of local homogeneity is necessary, however, to allow the measurements to be made at different locations within the beam cross-section.

To test for anisotropy, a series of evaluations must be made at different orientations of $\vec{\rho}$ until a set of perpendicular axes is found with respect to which the degree of anisotropy, if present, is a maximum. These will probably be found to be the horizontal and vertical directions. Any significant anisotropy should also be apparent from observation of the shape of the intensity pattern, if the conclusions of the NBS experiment described on page 80 are valid. According to this report, the discrete elements of the pattern should be stretched out in one direction as a result of anisotropic effects of the atmosphere.

Finally, it should be noted that several intensity pictures will probably be needed to provide a sufficient number of ensemble points to establish a single, and statistically significant ensemble average.

SUMMARY OF ERRORS

Sources of error in this experiment are

Resolution Error - the width of the image tube readout beam limits the resolution of the correlation function measurement to an amount $2D_{LL}(r_c)$, where the "characteristic length" r_c as defined in the Glossary and in this instance, relates to the size of the effective apertures due to the images at the Fresnel lens of the readout beam spot at the face of the image tube.

Amplitude Error - an uncertainty of magnitude ΔA in the measurement of the light amplitude A causes an error of magnitude.

$$\left(\frac{\Delta A}{A}\right)_{rms} \left\langle \log \frac{A}{A_o} \right\rangle \sqrt{\frac{2}{n-2}} \tag{4.1}$$

in the correlation function measurement, where n is the number of repeated measurements.

Receiver Noise Error - contributes to the measurement of $C_{II}(\vec{\rho})$ the error term

$$\sqrt{\frac{e}{4\eta A_o^2 \tau}} \left\langle \log \frac{A}{A_o} \right\rangle \sqrt{\frac{2}{n-2}} \tag{4.2}$$

where e = electronic charge

η = quantum efficiency of the photosurface of the image tube

τ = time constant of the receiver circuit

Background Light - produces an error which can be corrected by multiplying the measured value of $C_{II}(\vec{\rho})$ by the factor

$$\frac{P_s + P_b}{P_s} \tag{4.3}$$

where P_s and P_b are the signal and background powers, respectively.

Variance of the Correlation Function - this has the magnitude

$$C_{LL}^2(o) + C_{LL}^2(\rho) \tag{4.4}$$

The combined effect of the above errors can be expressed

$$\left(\frac{P_s + P_b}{P_s}\right)\left\{C_{LL}(\rho,\tau) + 2D_{LL}(r_c) \pm \sqrt{2\left[\left\langle\left(\frac{\Delta A}{A}\right)^2\right\rangle + \frac{e}{4\eta A_o^2 \tau}\right]\left\langle\log\frac{A}{A_o}\right\rangle^2 + C_{LL}^2(o) + C_{LL}^2(\rho)} \cdot \frac{1}{\sqrt{n-2}}\right\}$$

Accuracy Discussion

To aid in forming a picture of the effects being measured in this experiment, the following brief discussion of an actual observation is presented. A horizontal laser propagation experiment was conducted at the National Bureau of Standards (References 6, 7) using a He-Ne gas laser (λ = 6328Å) with a 4.5 inch transmitting reflector. The beam diameter at a prescribed distance was minimized by focus adjustments. Under ordinary atmospheric conditions, the beam diameter at a distance of 9 miles corresponded roughly to a 10 arc second beam divergence. The detailed structure, as seen on a ground glass screen, consisted of illuminated fragments averaging roughly 8 by 3 inches, stretched out horizontally. The sizes, shapes and numbers of fragments appeared to change over periods on the order of fractions of a second. It was noted that the magnitude of the beam wander appeared to be much less than the value of several beam diameters reported earlier. As observed visually, the beam appeared to remain rather stationary in position.

Because of the discrete structure of the Fresnel lens used in the present experiment, a ring structure is superimposed on the image of the laser formed on the faces of the image tubes. Since this spurious pattern is a hindrance in determining the true structure of the beam, its effect on the measurement must be eliminated by designing the receiver so that the width of the image tube readout beams are several times the spatial period of the Fresnel structure at the tube faces. The optimum width for this purpose would be $(n + \frac{1}{2})$ times the period distance (n = integer), neglecting the effects of intensity fluctuations. In general, however, the intensity at a given point as measured by the image tube is subject to some degree of error, and the corresponding light amplitude A_i at that point will be uncertain by an amount ΔA_i. This uncertainty can be taken to include the effect of variations in sensitivity over the face of the tube. The magnitude of ΔA_i can be formed by illuminating the Fresnel lens uniformly and observing the variations in the output over the surface of the tube. For a high quality vidicon or image orthicon, however, this effect should be small.

ERROR ANALYSIS

Resolution Error

Since the irradiance varies from point to point on the laser wavefront, the intensity sensed by the image tube is an averaged value of that portion of the wavefront "seen" by the readout beam, and as a result, the derived values of the structure functions and correlation functions are somewhat uncertain. This uncertainty can be expressed as the structure function $2\ D_{LL}(r_c)$ for a characteristic distance r_c, the factor 2 being due to the combined effect of the two effective apertures which are the images at the Fresnel lens of the readout beam spot on the face of the image tube.

Amplitude Error

As stated above, the measurement of the amplitude will be uncertain by an amount ΔA_i . The effect of this error on the computation of the log amplitude correlation function is described by the following equation:

$$\left\langle \left(\log \left(\frac{A_1}{A_o} \right) + \frac{\Delta A_1}{A_1} \right) \left(\log \left(\frac{A_2}{A_o} \right) + \frac{\Delta A_2}{A_2} \right) \right\rangle$$

$$= \left\langle \log \left(\frac{A_1}{A_o} \right) \log \left(\frac{A_2}{A_o} \right) \right\rangle + \left\langle \frac{\Delta A_1}{A} \log \left(\frac{A_2}{A} \right) \right\rangle + \left\langle \frac{\Delta A_2}{A} \log \left(\frac{A_1}{A_o} \right) \right\rangle + \left\langle \frac{\Delta A_1}{A} \frac{\Delta A_2}{A} \right\rangle$$

$$= C_{LL}(\rho) + \left\langle \frac{\Delta A_1}{A_1} \right\rangle \left\langle \log \frac{A_2}{A_o} \right\rangle + \left\langle \frac{\Delta A_2}{A_2} \right\rangle \left\langle \log \frac{A_1}{A_o} \right\rangle + \left\langle \frac{\Delta A_1}{A_1} \right\rangle \left\langle \frac{\Delta A_2}{A_2} \right\rangle$$

In the last expression use is made of the fact that the uncertainties in ΔA_1 , ΔA_2 , are independent, zero-mean random variables, so that all terms but the first are zero. Hence, the conclusion is that the averaging process is unaffected by the errors in A_i . However, the uncertainty of $C_{LL}(\rho)$ does depend on the number of measurements n used to calculate it. From statistical theory, the rms error term, which is proportional to the standard deviation of the amplitude is easily found to be

$$\left(\frac{\Delta A}{A}\right)_{rms} \left\langle \log \frac{A}{A_o} \right\rangle \sqrt{\frac{2}{n-2}} \qquad (4.1)$$

Receiver Noise Error

When the presence of receiver noise is taken into account, the analysis is similar to the above calculation with ΔA_i , and leads to the same conclusion, i.e., that the average value of the correlation function is not affected (as long as the noise level is reasonably small). The magnitude of ΔA_n arising from the noise depends on the time constant τ of the receiver circuit in the following way. If i_o is the average value of the output current i, e is the electronic charge and $i = \eta A^2$ where η is the quantum efficiency of the photosurface and A^2 is the intensity of the incident light, then

$$\frac{\langle (i-i_o)^2 \rangle}{i_o^2} = \frac{e}{i_o \tau}$$

$$\frac{\langle (\Delta A_n{}^2)^2 \rangle}{A_o^4} = \frac{e}{\eta A_o^2 \tau}$$

$$\left(\frac{\Delta A_n}{A_o}\right)_{rms} = \sqrt{\frac{e}{4 \eta A_o^2 \tau}}$$

and the standard deviation of C_{LL} due to receiver noise becomes

$$\sqrt{\frac{e}{4 \eta A_o^2 \tau}} \qquad \left\langle \log \frac{A}{A_o} \right\rangle \qquad \sqrt{\frac{2}{n-2}} \tag{4.2}$$

Background Light

When the background light is not negligible (but still small in comparison to the signal strength), the measured correlation function can be corrected to its true value in the absence of background by multiplying by the factor

$$\frac{P_s + P_b}{P_s} \tag{4.3}$$

where P_s and P_b are proportional to the signal and background power, respectively.

Variance of Correlation Function

Finally, the variance associated with the evaluation of the log amplitude correlation function was shown in Experiment No. 3 to be equal to

$$C_{LL}^{2}(0) + C_{LL}^{2}(\rho) \qquad (4.4)$$

SPECTRAL SPREADING MEASUREMENT - EXPERIMENT NO. 5

OBJECTIVE

To measure the frequency spreading Δf of laser light transmitted through the atmosphere, resulting from fluctuations in the refractive index of the air along the transmission path.

EXPERIMENTAL PROCEDURE

Description

The transmitted laser beam, as shown in Figure 7, is split into two beams of equal intensity which are transmitted over paths having equal lengths and which are sufficiently separated from each other so that there is no correlation between the refractive indexes along the two beams. By use of a frequency modulator, the frequency of one of the beams is shifted by an amount ω_a. After reflection from identical corner reflectors, the two returning beams are combined and focused on a single PM tube. The resulting beat frequency between ω_a and Δf is measured with a conventional frequency analyzer.

Discussion

The phase of the PM output is given by $\omega_a t + \phi_{21}$ (see the discussion of Experiment 2), with the corresponding frequency $\omega_a + \dot{\phi}_{21}$. $\dot{\phi}_{21}$ is a random variable which describes the instantaneous frequency shift Δf caused by the refractive index fluctuations of the atmosphere along the transmission path. Specifically, the variance σ_f^2 of Δf is calculated from the measured frequency fluctuations according to the relation

$$\sigma_f^2 = \langle \dot{\phi}_{21}^2 \rangle$$

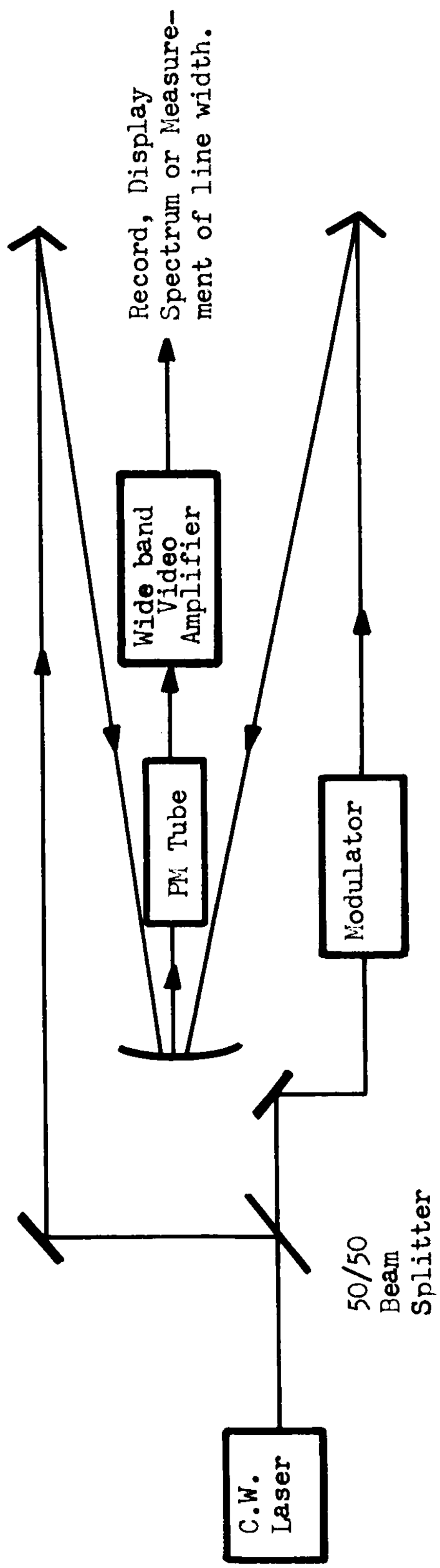

Figure 7. Measurement of Turbulence Induced Spectral Spreading.

To determine the effect of laser frequency drift on the determination of Δf , a measurement should be made with one of the two paths slightly shorter than the other. If the variances $\langle f^2 \rangle$ of the frequency spread measured for different values of ΔD_i are plotted on the same graph with the curve for $\Delta D = 0$ then the vertical separation $\langle \Delta f^2 \rangle$ between the $\Delta D = 0$ curve and the curve for ΔD_i gives the variance of the laser frequency drift during the time interval $\Delta t_i = \Delta D_i/c$ corresponding to the difference ΔD_i . A plot of these values then yields a curve of the variance of the laser frequency drift as a function of time.

SUMMARY OF ERRORS

Factors affecting the accuracy of the frequency spreading measurement include

1. Path Distance - the dependence of the frequency spread ϕ_{21} on the length of the light path is expressed in the relation

$$\langle \phi_{21}^2 \rangle \sim 2(2D - D_c + 2D_r) \tag{5.1}$$

where 2D is the round-trip geometrical path length from the transmitter to the corner reflector and back to the receiver, D_c is the segment of path in front of the PM tube which is common to the two beams, and D_r is a path distance in the region of the corner reflector where the beam passes through the same (and therefore correlated) refractive index fluctuations twice. Since D_r is indeterminate, the path ambiguity must be avoided by either blocking out the center (correlated) region of the corner reflector, or else using separate plane mirrors in place of the corner reflectors, depending on the experiment geometry. A test measurement to confirm this hypothesis is recommended before undertaking either of these modifications, however.

2. Amplitude Fluctuations - these give rise to a spurious power density spectrum which could obscure or degrade the frequency spreading measurement. This effect is expected to be small, however.

3. Laser Frequency Drift - can be determined from measurements of the frequency spread for equal paths and unequal paths, according to the relation

$$\left\langle f^2 \right\rangle_{\text{drift}} = \left\langle (f - f_a)^2 \right\rangle_{\text{unequal}} - \left\langle (f - f_a)^2 \right\rangle_{\text{equal}} \qquad (5.2)$$

4. Receiver Noise - adds to the power spectrum $P(\omega)$ of the received signal the term

$$2q^2 \eta \bar{n} B_o$$

where q = electronic charge

η = quantum efficiency of the PM tube

$\bar{n}$ = number of photons/sec of frequency ν entering the receiver

B_o = bandwidth of the receiver (cps)

ACCURACY DISCUSSION

The effects of various experimental factors on the accuracy of the measurement are briefly described in this section.

In regard to errors in the frequency fluctuation, no special precautions are required in setting up the experiment. Stable frequency modulation sources are available, while errors in the alignment of the equipment affect only the power levels, not the frequencies. Finally, frequency drifting of the laser is eliminated by making the two light paths equal in length.

One factor which affects the power levels is the size of the receiver aperture. The amplitude of the fluctuations resulting from atmospheric effects decreases with increasing size of the aperture.

Probably the most difficult task encountered in setting up this experiment is that of accurately determining the path length. The segment of the light path lying in front of the PM tube is common to both light beams and hence induces no phase difference fluctuation between the two beams. In the vicinity of the corner reflector, however, conditions are more complicated. As mentioned in Volume I of the Task II report, (Reference 1) the variations in the index of refraction from point to point are not random over the short paths traversed by the light undergoing multiple reflections at the corner reflector. The corresponding frequency fluctuations therefore depend on this path distance, D through the relation $\sqrt{2D}$, rather than $\sqrt{D}$, as for the remainder of the laser path. Specifically, if the geometrical path length from the transmitter to the corner reflector and back to the PM tube is 2D , while D_c is the path in front of the PM tube, and D_r is the path in the region of the corner reflector where the beam passes through the same density (index of refraction) distribution twice, then the total optical path length can be written

$$2D - D_c - 2D_c + 4D_r = 2D - D_c + 2D_r$$

where the term $4D_r$ accounts for the fact that the light passes twice through an area having the same atmospheric disturbance.

Then the dependence of the frequency fluctuation for both paths becomes

$$\langle \phi_{21}^2 \rangle \sim 2(2D - D_c + 2D_r) \tag{5.1}$$

Since D_r is undetermined, an irremovable systematic error is introduced into the results when a corner reflector is used in the experiment under the conditions described.

At least two possible remedies for this problem are available. First, if the width of the corner reflector is several times greater than the correlation distance of the refractive index fluctuations (which might be around a few centimeters, depending on the range and choice of path between the transmitter and corner reflector), and if the receiver aperture lies within the near field region of the beam emerging from the corner reflector (given approximately by d^2/λ , where d is the diameter of the corner reflector), then the indeterminacy in D can be resolved by blocking out the center region of the reflector to a radius greater than this correlation distance. By this process, the corner reflector becomes, in effect, an ordinary array of separate plane mirrors for which simple ray optics considerations are sufficient.

Of course, the experiment cannot always be scaled in such a way that the above conditions obtain, and even when they do, the simple modification described may not suffice to avoid the indeterminacy if more than one reflector is required. In such cases, the advantages of using corner reflectors may have to be sacrificed in favor of an arrangement of large, plane mirrors (of good optical quality), such as that depicted in Figure 8, which completely eliminates these complications, at the cost of some extra effort needed to ensure proper alignment of the mirrors.

Before resorting to plane mirrors, however, a series of actual test measurements with a corner reflector place at several distances should be made to confirm this hypothesis. If the deviation from the $\sqrt{D}$ law turns out to be unmeasureable, then the experiment can be considerably simplified, with no loss in accuracy, by using corner reflectors.

ERROR ANALYSIS

Amplitude Fluctuations

The output of the PM tube, given by

$$\tfrac{1}{2}\left(A_1^2(t) + A_2^2(t)\right) + A_1(t)A_2(t)\cos(\omega_a t + \phi_{21})$$

is centered around the modulator frequency ω_a. Random fluctuation in the phase difference ϕ_{21} induce a frequency spreading of magnitude Δf which is proportional to the time rate-of-change ϕ_{21}. Δf is therefore a random variable which is measured by determining its spectral density distribution. But, as indicated above, the amplitudes A_1 and A_2 of the two portions of the laser beam also vary with time, and thus also give rise to a power density spectrum which is equal to the cosine transfer of the auto-correlation

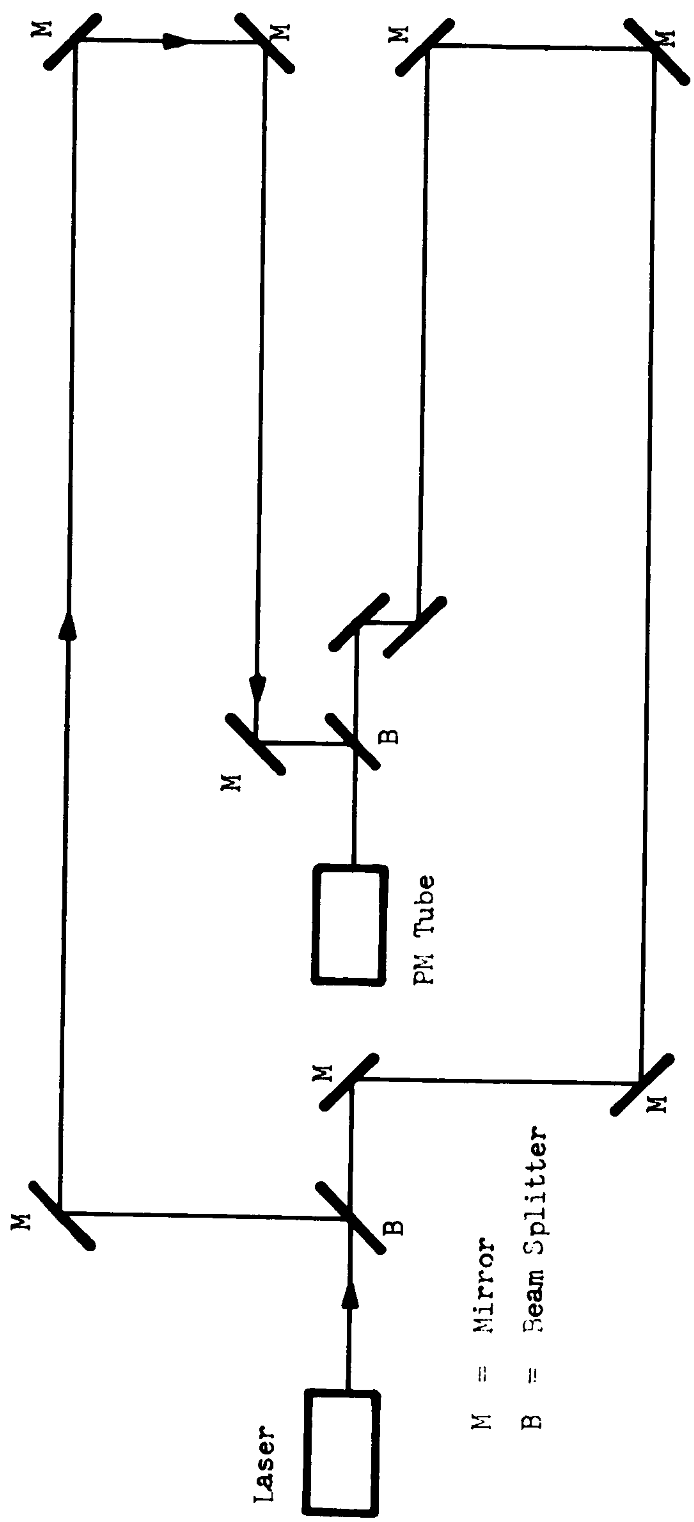

Figure 8. Alternate Configuration for Spectral Spreading Measurement.

function of the amplitude product. Indications are that the frequency spreading due to the amplitude fluctuations is much smaller than the spreading caused by the phase fluctuations, so that the effect of the former can be disregarded. This assumption should, however, be verified by comparing the results of Experiment No. 6 (Power Fluctuation Measurement).

Laser Frequency Drift

The variance of the frequency spread, $\left\langle (f - f_a)^2 \right\rangle$ is determined by the turbulent air, and when the two path lengths are unequal, it is also affected by the average drift of the laser frequency during the time $\Delta t = \Delta D/c$ associated with the path difference ΔD (c is the velocity of light). To measure the laser drift, the frequency spread must first be measured for equal path lengths, following which a series of measurements for several values of ΔD is made. The rms laser drift frequency is then obtained by subtracting the variance for the equal path measurement from the value for the unequal path:

$$\left\langle f^2 \right\rangle_{drift} = \left\langle (f - f_a)^2 \right\rangle_{unequal} - \left\langle (f - f_a)^2 \right\rangle_{equal} \quad (5.2)$$

Noise

Finally, the effect of noise in the receiver channel is to add a term $2q^2 \eta \bar{n} B_o$ to the power spectrum of the received signal, where

q = electronic charge

η = quantum efficiency of the PM tube

$\bar{n}$ = number of photons/sec of frequency ν entering the receiver

B_o = bandwidth of the receiver (cps)

POWER FLUCTUATION MEASUREMENT - EXPERIMENT NO. 6

OBJECTIVE

To measure the received light power and its fluctuations as a function of the aperture size of the receiver.

EXPERIMENTAL PROCEDURE

Description

As shown in Figure 9, the incoming laser light is captured with a receiver of large and variable aperture. The light power P is measured with a PM tube, amplified and recorded for later evaluation of $\langle F \rangle$, $\langle P^2 \rangle$ and the power (frequency) spectrum $P(\omega)$. The results of the experiment can be compared with the theoretical values given in Task I, Volume 2 (Reference 8).

The entrance aperture has sun shields and light baffles to prevent any stray light from entering the PM tube. The entrance aperture can be changed by using a series of discs of varying diameters. Provision must be made to calibrate the PM tube with a standard light source. Finally, the recorder must have sufficient bandwidth to accomodate communication frequencies (bit rates).

Discussion

As will be explained in detail later under ACCURACY DISCUSSION for this measurement, the experiment should be performed under conditions when $C_{LL}(\rho)$ is known, either by establishing a correlation with meteorological conditions existing at the time of a previous measurement, followed by a spot check of $C_{LI}(\rho)$, or by a complete re-determination of $C_{LL}(\rho)$ at the time of the experiment. In the former case, a pair of counter-rotating discs with pinhole

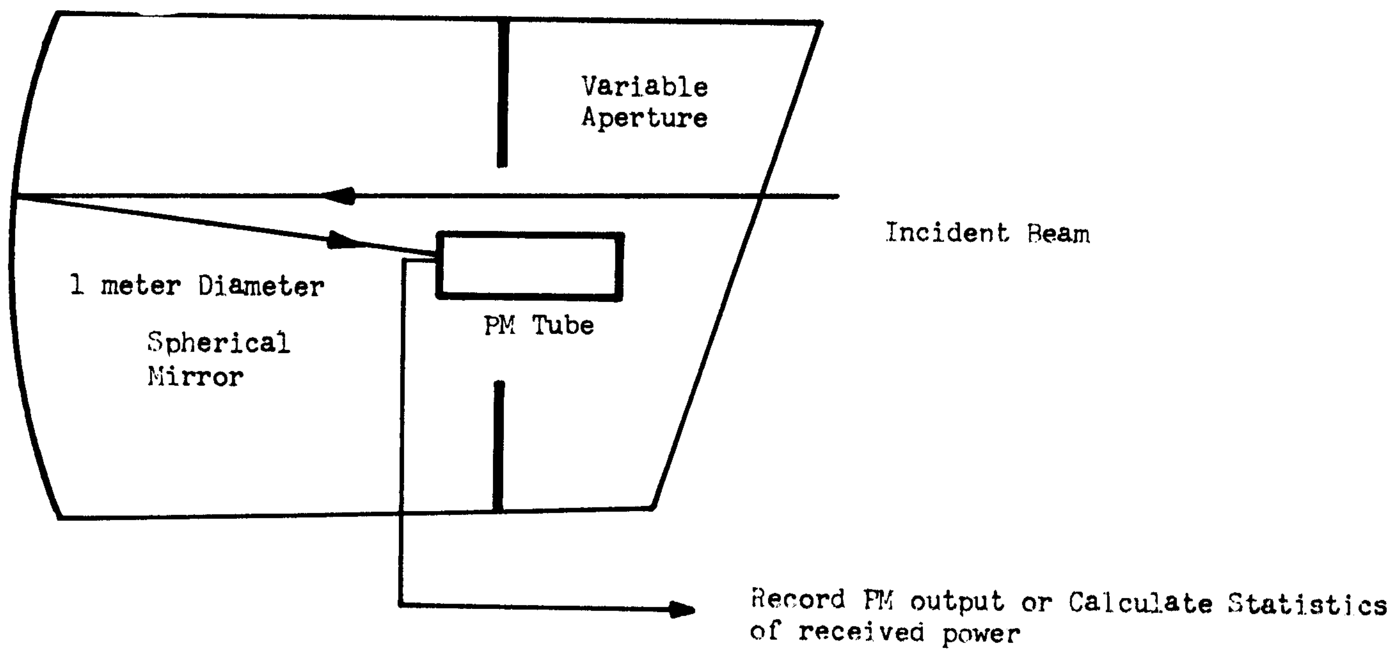

Figure 9. Measurement of the Power Fluctuation

apertures is suggested for making the spot check. A secondary experiment is recommended to determine whether or not the total beam power fluctuates in time. For this measurement, the receiver aperture must be greater than the total width of the beam.

Except when the effects of background light are being assessed, this experiment is best performed at night.

SUMMARY OF ERRORS

The variance for a single measurement of the power fluctuation is the sum of the following two components

Relative Variance of the Power Fluctuations - given by

$$\sigma_s^{\,2} = \frac{\langle P_m^2 \rangle - \langle P_m \rangle^2}{\langle P_m \rangle^2} \left(\frac{\langle P_m \rangle^2}{\langle P_s \rangle^2} \right) \qquad (6.3)$$

where P_m is the total measured power and P_s is the true signal (laser) power.

Signal-to-Noise - given by

$$- \frac{1}{2(S/N)R} \qquad (6.82)$$

where (S/N) is the signal-to-noise ratio at the receiver, and R is the ratio of total photons/sec to signal photons/sec. In the frequency domain, the noise error term to be added to the power spectrum $P(\omega)$ is

$$- \frac{1}{2(S/N)R} \quad \frac{B_a}{B_0} \qquad (6.92)$$

where B_0 and B_a are the bandwidths of the receiver and the frequency analyzer, respectively.

The total variance in the value of the power fluctuation determined from n measurements is

$$\pm \sigma_s{}^2 \sqrt{\frac{2}{n-1}}$$

where $\sigma_s{}^2$ is the sum of A_1 and A_2

ACCURACY DISCUSSION

In this experiment, it is important that the value of the log amplitude correlation function $C_{II}(\rho)$ be known at the time of the measurement, because correct interpretation of the experimental results hinges on comparison with a theoretical model which is based on this function. Although a concurrent measurement of $C_{II}(\rho)$ is necessary for a strictly rigorous determination of the intensity fluctuations, the additional experimental complications involved prompt search for ways of getting around this requirement, even at the cost of some loss of rigor in the result.

A suitable alternative is the following. Since the log amplitude correlation function is dependent on the local atmospheric conditions prevailing at the time of the measurement, the chances of the same $C_{II}(\rho)$ occurring are greatest when the atmospheric conditions are similar. Thus, it may be possible to obviate the need for a complete repeated measurement of the correlation function by choosing a day on which the conditions are similar to those existing at the time $C_{II}(\rho)$ was measured previously, making the simple spot check described below, and if the check evidences the same $C_{II}(\rho)$, then proceeding with the experiment. If the previous measurements of $C_{LL}(\rho)$ were made several times under a variety of conditions, the chances of one of these "data points" repeating on a given day will be improved. Of course, the two experiments must be carried out over the same optical link to eliminate variations due to change of location. In this connection, it should be noted that the value of $C_{LL}(\rho)$ cannot be extrapolated from one set of environmental conditions to another, since there is no way of knowing which of the several possible parameters (or combinations of parameters) caused the change, or the extent of the resulting change. Hence, if a correlation with previous measurements cannot be found, the only recourse is to completely re-determine $C_{LL}(\rho)$ at the time (and location) of the experiment.

A spot check of $C_{LL}(\rho)$ can be easily made with the aid of a pair of disks having identical pinholes at the same radial distance $\rho/2$, the disks being placed coaxially near the plane of the iris (aperture stop). By counter-rotating the disks at a constant angular velocity (e.g. with a constant speed motor), the incoming light will be transmitted once per revolution from each of two points separated by a distance ρ. From the data thus obtained $C_{LL}(o)$ and $C_{LL}(\rho)$ can be evaluated and compared with the previous values. If a correlation with one of these is found, then that value may be assumed to hold for the experiment. The same check could also be made by masking the entire aperture except at two points separated by a distance ρ and alternately measuring the light from each by means of a strategically placed rotating disc which alternately cuts off each of the rays.

An additional point of interest is to determine if the total power in the beam remains constant (with time varying spatial distribution), as has been implicitly assumed in the analysis, or whether, as reported recently, (Reference 9) the total power fluctuates in time. (Such an effect could be due to longitudinal variations in air density along the transmission path.) This check can be easily made by designing the experiment geometry so that the receiving aperture is larger than the total beam. If reflecting optics are used, allowance must be made for the small portion of the beam which is obscured by the center reflector.

ERROR ANALYSIS

Power Fluctuation

The instantaneous measured power P_m is comprised of both the signal power P_s and the background power P_b:

$$P_m = P_s + P_b$$

$$\text{or} \qquad P_s = P_m - P_b \tag{6.1}$$

From this relation it follows that

$$\langle P_s^2 \rangle = \langle P_m^2 \rangle - 2\langle P_m \rangle \langle P_b \rangle + \langle P_b^2 \rangle \tag{6.21}$$

$$\langle P_s \rangle^2 = \langle P_m \rangle^2 - 2\langle P_m \rangle \langle P_b \rangle + \langle P_b \rangle^2 \tag{6.22}$$

Since the background power is due to an extended, incoherent source, while the laser is effectively a coherent point source, the fluctuations in P_b will be negligible in comparison to the fluctuations in P_s , so $\langle P_b^2 \rangle - \langle P_b \rangle^2 \simeq 0$

and the relative variance $\sigma_s{}^2$ can be written, using Equations 6.21 and 6.22,

$$\sigma_s{}^2 = \frac{\langle P_s{}^2\rangle - \langle P_s\rangle^2}{\langle P_s\rangle^2} = \frac{\langle P_m{}^2\rangle - \langle P_m\rangle^2}{\langle P_s\rangle^2}$$

$$= \frac{\langle P_m{}^2\rangle - \langle P_m\rangle^2}{\langle P_m\rangle^2} \; \frac{\langle P_m\rangle^2}{\langle P_s\rangle^2}$$

or

$$\sigma_s{}^2 = \frac{\langle P_m\rangle^2}{\langle P_s\rangle^2} \qquad (6.3)$$

since $\sigma_m{}^2 = \dfrac{\langle P_m^2\rangle - \langle P_m\rangle^2}{\langle P_m\rangle^2}$.

Now $\sigma_m{}^2$ and $\langle P_m\rangle^2$ can be calculated from the recorded measurements of P_m , while $\langle P_s\rangle^2$ can be evaluated from Equation 6.22 , where $\langle P_b\rangle$ can be measured by simply turning off the laser transmitter ($\langle P_s\rangle = 0$) . Hence, by means of Equation 6.3 , the (normalized) fluctuations in signal power due to the atmosphere can be determined.

Signal-to-Noise

The light intensity being recorded in this experiment is proportional to the output of the photomultiplier which contains, in addition to the desired signal, a random component due to the shot noise originating in the PM itself. This current fluctuation is given by

$$\sigma_i^2 \equiv \frac{\langle (i - i_s)^2 \rangle}{i_s^2} = \frac{q}{i_s \tau} = \frac{1}{\eta \bar{n}_\tau} \tag{6.4}$$

where σ_i^2 is the relative variance of the output current i with respect to the average current i_s , q is the electronic charge, η is the quantum efficiency of the photosurface (electrons/photon), and $\bar{n}$ is the average number of photons received during the sampling time τ . The latter is effectively the time constant of the receiver circuit. In this expression i_s is assumed to remain substantially constant during the sampling period τ . Actually, i_s is not a constant, but varies slowly over a characteristic period T which is much greater than τ . Since τ is a design parameter, it can be made great enough so the above stability condition on i_s is assured, without at the same time being so long as to "wash out" the intensity fluctuations being measured.

If the long term $(t \gg T)$ average of i_s is denoted by i_o , i.e. $\langle i_s \rangle = i_o$ then, using the fact that the output current i_s is proportional to the received power P_s which produced it, the variance of i_s can be written

$$\sigma_{is}^2 = \frac{\langle (i_s - i_o)^2 \rangle}{i_o^2} = \frac{\langle i_s^2 \rangle - i_o^2}{i_o^2} = \frac{\langle P_s^2 \rangle - \langle P_s \rangle^2}{\langle P_s \rangle^2} = \sigma_{ps}^2$$

The measured instantaneous power P_m is proportional to the total output current, i, which is comprised of both i_s and the shot noise current $(i - i_s)$. Hence, using the defining relation $\langle i - i_s \rangle = 0$, together with Equation 6.4,

$$\frac{\langle P_m^2 \rangle}{\langle P_m \rangle^2} = \frac{\langle i^2 \rangle}{i_o^2} = \frac{\left\langle \left[i_s + (i - i_s) \right]^2 \right\rangle}{i_o^2} = \frac{\langle i_s^2 \rangle + \langle (i - i_s)^2 \rangle}{i_o^2}$$

$$= \frac{1}{i_o^2} \left(\langle i_s^2 \rangle + \langle i_s \rangle^2 \frac{q}{i_s \tau} \right)$$

$$= \frac{\langle P_s^2 \rangle}{\langle P_m \rangle^2} + \frac{q}{i_s \tau} \frac{\langle P_s \rangle^2}{\langle P_m \rangle^2}$$

which can also be written

$$\frac{\langle P_m^2 \rangle - \langle P_m \rangle^2}{\langle P_m \rangle^2} = \frac{\langle P_s^2 \rangle + \frac{q}{i_s \tau} \langle P_s \rangle^2 - \langle P_m \rangle^2}{\langle P_m \rangle^2}$$

or, after rearranging

$$\frac{\langle P_s^2 \rangle - \langle P_m \rangle^2}{\langle P_m \rangle^2} = \frac{\langle P_m^2 \rangle - \langle P_m \rangle^2}{\langle P_m \rangle^2} - \frac{\langle P_s \rangle^2}{\langle P_m \rangle^2} \frac{q}{i_s \tau} \qquad (6.5)$$

But

$$\langle P_m - P_s \rangle = 0, \quad \langle P_m \rangle = \langle P_s \rangle$$

so Equation 6.5 becomes

$$\frac{\langle P_s^2 \rangle - \langle P_s \rangle^2}{\langle P_s \rangle^2} = \frac{\langle P_m^2 \rangle - \langle P_m \rangle^2}{\langle P_m \rangle^2} - \epsilon$$

where

$$\epsilon = \frac{q}{i_s \tau} = \frac{1}{\eta \bar{n}_\tau} \tag{6.6}$$

The "error term" of Equation 6.6 can be expressed in a more meaningful form by introducing signal-to-noise (S/N) considerations. The applicable S/N expression in the absence of background noise is

$$\frac{S}{N} = \frac{\eta \bar{n}_s}{2Bo} \tag{6.71}$$

where the noise in this case is the shot noise of the receiver.

With background noise present,

$$\frac{S}{N} = \frac{\eta \bar{n}_s}{2Bo(\bar{n}_\tau / \tau)} \tag{6.72}$$

where η = quantum efficiency of the PM

$\bar{n}_s = \bar{P}_s / n\nu$ = number of signal photons/sec of frequency ν entering the receiver

$\bar{n}_\tau$ = total number of photons entering receiver in time τ,

i.e., $\bar{n}_\tau = \bar{n}_s \tau$

τ = sampling time

B_o = receiver bandwidth (cps)

Making the substitution $\bar{n}_s = \bar{n}_\tau/\tau$ and assuming the low-pass filter condition $B_o\tau \approx 1$ Equation 6.6 can be written with the help of Equation 6.71,

$$\epsilon = \frac{1}{2(S/N)} \qquad \text{(no background)} \qquad (6.81)$$

Similarly, letting $\bar{n}_\tau/\tau = R\bar{n}_s$ for the background case, where R is the ratio of total photons to signal photons, Equation 6.72 can be written

$$\frac{S}{N} = \eta\frac{(\bar{n}_\tau/\tau)^2 \ (1/R^2)}{2B_o\bar{n}_\tau/\tau} = \frac{\eta\,\bar{n}_\tau}{2R^2} \qquad (B_o\tau = 1)$$

or

$$\bar{n}_\tau = (S/N)\ \frac{2R^2}{\eta}$$

Hence, the error term becomes, for this case,

$$\epsilon = \frac{1}{\eta\,\bar{n}_\tau} = \frac{1}{2(S/N)R^2} \qquad \text{(background present)} \ (6.82)$$

which reduces to Equation 6.81 when $R = 1$, i.e. when the signal flux equals the total flux.

Power Spectrum

The bandwidth B_a of the frequency analyzer used in evaluating the power spectrum of the signal fluctuations must be much narrower than the low pass filter bandwidth B_o of the receiver. When the signal-to-noise ratio is calculated by means of the frequency analyzer, the following equations, corresponding to Equations 6.71,2 and 6.81,2 apply:

$$\left(\frac{S}{N}\right)_{\omega} = \frac{\eta \bar{n}_{\tau}}{2R^2} \frac{B_o}{B_a} \qquad \text{(no background)} \qquad (6.91)$$

$$\left(\frac{S}{N}\right)_{\omega} = \frac{\eta \bar{n}_s^2}{2B_o(\bar{n}_{\tau}/\tau)} \frac{B_o}{B_a} \qquad \text{(no background)} \qquad (6.92)$$

and

$$\epsilon_{\omega} = \frac{\langle P_m^2 \rangle}{\langle P_m \rangle^2 \eta \bar{n}_{\tau}} = \frac{\langle P_m^2 \rangle}{\langle P_m \rangle^2 2(S/N)R^2} \frac{B_a}{B_o} \qquad (6.10)$$

When the power spectrum $P_m(\omega)$ of the measured power is calculated, the result will include a component of magnitude $2\eta q^2 B_a(\bar{n}_{\tau}/\tau)$ due to the receiver noise, and this term must be subtracted from the total to obtain the true signal power spectrum $P_s(\omega)$.

The order of magnitude of the fluctuation frequencies (power spectrum) to be expected can be inferred by comparing known star scintillations which have frequencies ranging up to 1000 cps, with a mean value around 100 cps.

HETERODYNE EQUIVALENT MEASUREMENT SYSTEM - EXPERIMENT NO. 7

OBJECTIVE

To measure the average irradiance at the center of the image of a focused laser beam, as a function of the diameter of the entrance aperture.

EXPERIMENTAL PROCEDURE

Description

The transmitted laser beam, Figure 10, is received with a collector of large and variable aperture. The diffraction pattern appearing at the focus of the collector is magnified by a field lens of high power and imaged in the plane of a pinhole which forms the aperture of a PM tube. The optical design is such that the width of the pinhole is less than about 10 percent that of the image, so that only the light flux contained in a small area around the center of the image is measured by the PM tube. After performing the calibration measurement (described under "MEASUREMENT ANALYSIS") using a local standard source (which can be the laser transmitter itself), the received power is measured for a series of receiver apertures of varying but known sizes. The size of the pinhole must also be accurately known.

To average out the intensity fluctuations caused by jitter (angular fluctuations) of the image, the time constant of the receiver circuit should be on the order of several seconds.

Discussion

The theory discussed in Reference 1 asserts that the heterodyne efficiency of an optical heterodyne detector is proportional to the ratio of the average flux density at the center of a focused laser beam in the presence

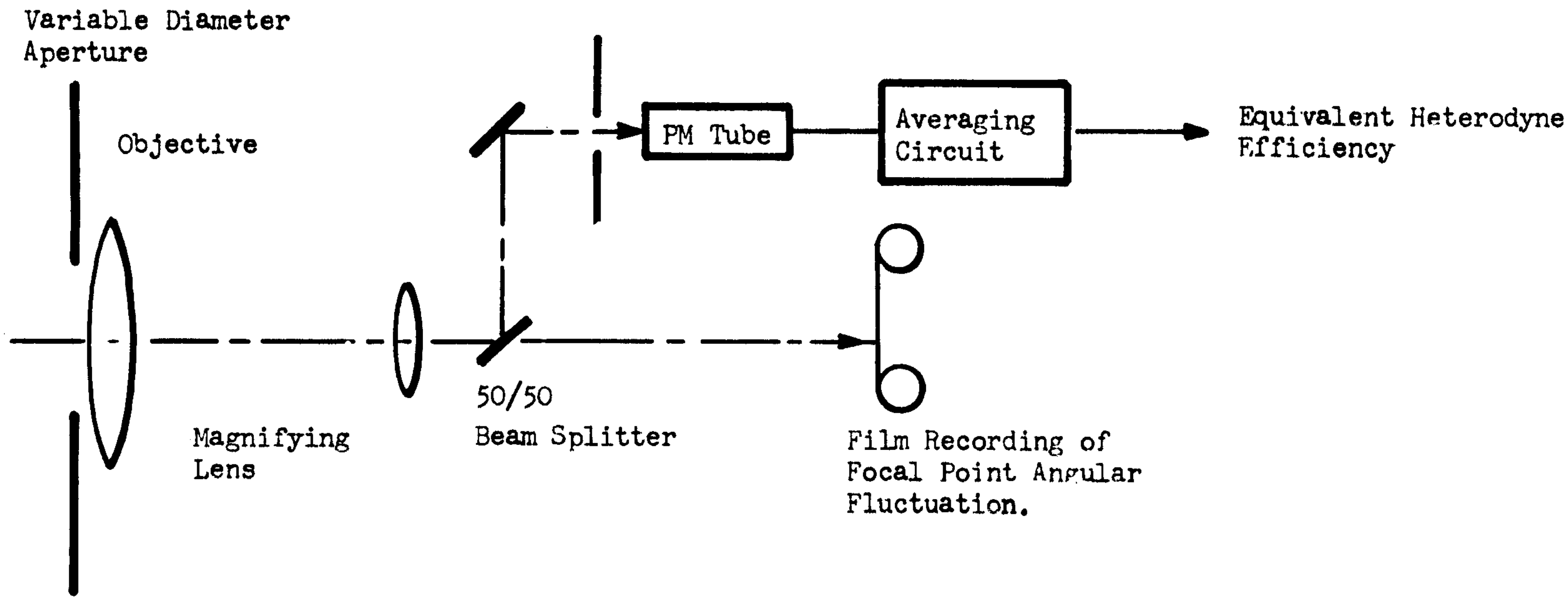

Figure 10. Measurement of Heterodyne Equivalent and Angular Fluctuation.

of optical abberations to the same quantity in the absence of abberations (the "Strehl definition"). By measuring this ratio with a diffraction-limited optical system (including the laser optics), the effects of atmospheric "abberations" alone can be extracted and compared with theoretical predictions.

SUMMARY OF ERRORS

Errors in the measurement can arise from the following experimental factors.

<u>Beamwidth</u> - must be sufficiently greater than the size of the receiver aperture to ensure that the local average irradiance does not vary significantly across the aperture.

<u>Pinhole Size</u> - must be accurately known and must be kept constant (free of dust, etc.) during the measurement.

<u>Ratio of Pinhole Diameter to Image Diameter</u> - must be less than 0.1 to minimize errors in determining the center irradiance H_{ro} of the image.

<u>Optics</u> - must be diffraction limited

<u>PM Tube</u> - must be operated in its linear range at all times. An attenuating filter should be used when measurements of total power P are made with the pinhole removed.

MEASUREMENT ANALYSIS

Irradiance Distribution of the Focused Beam

Since the irradiance of the focused laser beam is not uniform over the focal plane of the receiver (where the pinhole and PM tube are located), the value of $H_r = P_r/\pi r^2$, obtained by averaging the measured power P_r over the area of the pinhole is less than the true center irradiance H_{ro} required for the heterodyne equivalent measurement. The needed correction to H_r can be found by considering the irradiance distribution of an Airy disc, which describes the focused spot produced, according to theory, by an aberration-free optical system in the absence of atmospheric distortions.

If $H(\rho)$ denotes the irradiance of the Airy disc, then

$$H(\rho) = \left(\frac{2\,J_1(\rho)}{\rho}\right)^2 H_o \qquad [\text{power/area}] \tag{7.1}$$

where ρ is proportional to the radial distance from the center of the disc, J_1 is the first order Bessel function, and H_o, is the irradiance at the center of the disc. By integrating 7.1 , it can be shown (Reference 10) that the fraction $\bar{P}$ of the total power in the diffraction pattern which falls within a radium r measured from the center of the Airy disc is given by the expression

$$\bar{P}(x) = \int_0^x \left(\frac{2\,J_1(\rho)}{\rho}\right)^2 2\pi\rho\, d\rho = 1 - J_o^{\,2}(x) - J_1^{\,2}(x) \tag{7.2}$$

where J_o and J_1 are the zeroth and first order Bessel functions and the variable x is related to r by

$$x = (2\pi/\lambda)\, a\, (r/R_1) ,$$

with

$$\lambda = \text{wavelength} \tag{7.3}$$

$$a = \text{radius of the aperture}$$

$R_1 = Rf_1/f_2$ where R, f_1 and f_2 are the optical parameters defined in Figure 11 .

A sketch of $\bar{P}(x)$ is shown by the solid curve of Figure 11 ; the dotted curve shows the power received in the special case when the irradiance has

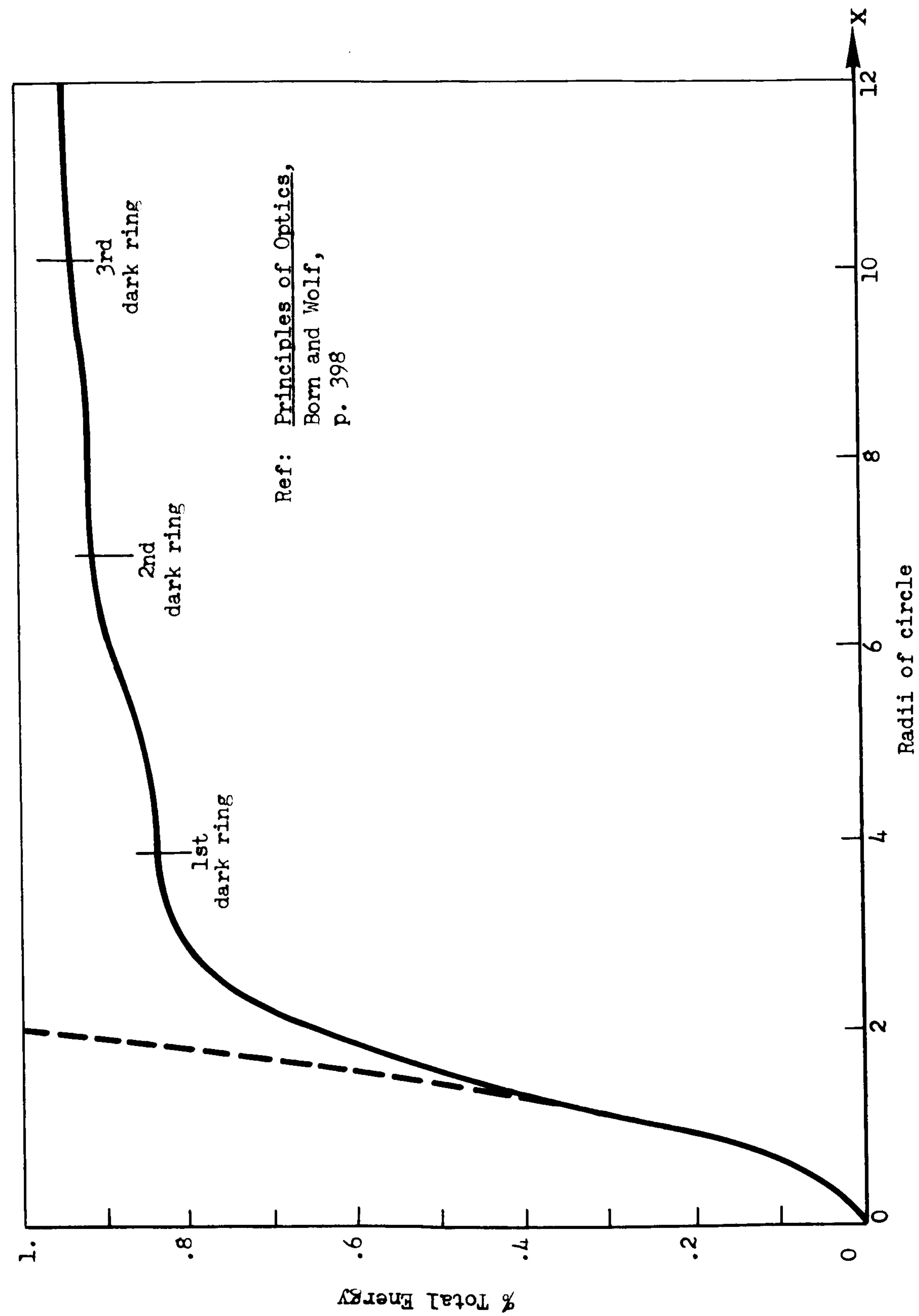

Figure 11. The Function $1 - J_0^2 - J_1^2(x)$ - Fraction of Total Energy Contained Within Circle in Fraunhofer Diffraction Pattern of a Circular Aperture

a constant value H_o over the entire extent of the pinhole. From consideration of these curves, along with Equation 7.2, it can be shown (Reference 10), that the irradiance H_o at the center of an Airy disc is related to the irradiance H_a averaged over a finite circular aperture symmetrically positioned at the center of the pattern, by the expression

$$H_o = H_a \frac{x^2/4}{1 - J_0^2(x) - J_1^2(x)} = H_a F_a(x)$$

or

$$H_o = \frac{P_a}{\pi r^2} F_a(x) \tag{7.41}$$

where

$$F_a(x) = \frac{x^2/4}{1 - J_o^2(x) - J_1^2(x)}$$

is the required correction factor.

The subscript a signifies that these relations pertain to a true Airy pattern. Actually, the distribution being measured includes the effects of atmospheric (and possibly optical system) distortions of the laser beam. However, since the aberrations induced by the atmosphere are random in direction, the disturbed intensity pattern sensed by the PM tube is also symmetrical and may be assumed (to first order of approximation) to be an enlarged Airy disc. Thus, letting $x' = \alpha x$, where $0<\alpha<1$, Equation 7.41 may also be used to relate H_{ro} to H_r, the actual measured irradiance:

$$H_{ro} = H_r F_a(x')$$

$$= \frac{P_r}{\pi r^2} F_a(x') \tag{7.42}$$

Taking the ratio of H_{ro} to the theoretical center irradiance H_o gives the Strehl definition $\mathscr{D}$, which as shown in Reference 1 is equal to the theoretical heterodyne efficiency of the receiver. From theory, the irradiance at the center of an Airy disc is (Reference 10)

$$H_o = \frac{PA}{\lambda^2 R_1{}^2} \quad \text{power/area} \tag{7.5}$$

where P is the total received power (measured with the pinhole removed), A is the receiver aperture, and R_1 is defined following Equation 7.3. Using Equations 7.3, 7.4, 7.5, the Strehl definition can then be written

$$\mathscr{D} \equiv \frac{H_{ro}}{H_o} = \left(\frac{P_r}{P}\right) \frac{\lambda^2 R_1{}^2}{\pi r^2 A} F_a(x') \tag{7.61}$$

or

$$\frac{H_{ro}}{H_o} = \left(\frac{P_r}{P}\right) \frac{4 F_a(x')}{x'^2} \tag{7.62}$$

Several values of $F_a(x')$ and $4F_a(x')/x'^2$ are tabulated here for reference:

x'	$F_a(x')$	$4 F_a(x')/x'^2$
0	1	
0.2	1	100
0.4	1.02	25.5
0.5	1.03	16.5
0.7	1.06	8.67
1.0	1.13	4.52

From the above tabulation and Equation 7.42, it is evident that if $x' < 0.4$, the measured irradiance differs from H_o by less than 2 percent and the power P_r entering the pinhole is about (1/25)th of the total power P.

The experimental significance of this condition can be determined from examination of Equations 7.1 and 7.2. Since J_1 (3.83) = 0, the radius r_o of the central spot is prescribed by the condition $x' = 2\pi a r_o/\lambda R_1 = 3.83 \simeq L$. Thus, $x' < 0.L$ implies the condition $r < r_o/10$ in which the pinhole diameter is less than about 10 percent of the spot size. By maintaining this condition (which is easily accomplished experimentally), the accuracy of the measurement is made relatively insensitive to the exact distribution of the focused image.

Equation 7.6 provides the basis for experimentally determining the ratio H_{ro}/H_o. First, the total power P received by the collecting aperture is measured with the pinhole removed. Then a similar measurement with the pinhole in place yields the value of P_r. The multiplying factor $L\ F_a(x')/x'^2$ of Equation 7.61 could in principle, be evaluated from measurements of the defining parameters. A better procedure, however, is to perform the following direct calibration measurement.

The factor $L\ F_a(x')/x'^2$ can be evaluated with the aid of a local coherent or quasi-coherent light source, which can be either the laser itself or another monochromatic point source of wavelength λ_c. From the way in which H_{ro} was defined, the ratio H_{ro}/H_o becomes unity when the light path does not pass through a turbulent atmosphere, provided the optical system is diffraction limited. If P_r and P are now measured with the local calibration source, Equation 7.61 gives

$$1 = \left(\frac{P_r}{P}\right)_c \frac{\lambda_c^2 R_1^2}{\pi r^2 A} F_a(x_c)$$

or

$$\frac{\lambda_c^2 R_1^2}{\pi r^2 A} = \left(\frac{P}{P_r}\right)_c \frac{1}{F_a(x_c)} \tag{7.7}$$

where the subscript c denotes the calibration values and the argument of F_a is x rather than x' because the focused spot in this case is a true Airy disc. The value of F_a can be determined from Equation 7.62 and the tabulation presented earlier with $H_{ro}/H_o = 1$. Allowing for the fact that the collection aperture $A = \pi a^2$ is a variable, while λ_c is not necessarily the same as λ, the laser wavelength, the Strehl definition (Equation 7.61) can be expressed in the form

$$\mathscr{D} \equiv \frac{H_{ro}}{H_o} = \left[\left(\frac{a\lambda_c}{a_c\lambda} \right)^2 \frac{F_a(x')}{F_a(x_c)} \left(\frac{P}{P_r} \right)_c \right] \frac{P_r}{P} \qquad (7.81)$$

If the laser transmitter (or another of the same frequency) is used for the calibration, and if as discussed above, the diameter of the pinhole is always kept less than 10% of the focused spot size, then $\lambda_c = \lambda$, $F_a(x')/F_a(x_c) \simeq 1$ and Equation 7.81 simplifies:

$$\mathscr{D} = \frac{H_{ro}}{H_o} = \left[\left(\frac{a}{a_c} \right)^2 \left(\frac{P}{P_r} \right)_c \right] \frac{P_r}{P}$$

Once the calibration has been performed, the experiment is carried out by measuring P_r and P for different values of the aperture a.

ERROR DISCUSSION

In this section the precautions necessary to minimize experimental inaccuracies will be discussed as they relate to the configuration of the optical elements and the performance constraints associated with the electronic components.

Optical Configuration

The first point of concern is to design the experiment in such a way that the width of the received beam is sufficiently greater than that of the collector aperture to ensure that the local average irradiance does not vary over the aperture. This condition, which is required by the plane wave assumption of the theoretical analysis, may be most easily verified by direct visual observation of the beam from the position of the aperture (A safety filter is advisable at close range to prevent possible eye injury). Of course, the beam must not be made so wide that the level of irradiance is reduced below a useable minimum.

A second design consideration is the size of the pinhole relative to the width of the focused image. As explained previously, the experiment is best performed with a pinhole which is not larger than about 10% of the image size, so that the function $F_a(x')$ of Equation 7.4 may be kept close to unity. If this limit is exceeded, an error of indeterminate magnitude will be incurred in the intensity measurement because of the fact that the actual irradiance distribution within the focused image is not known. It is also important that the pinhole be kept free of dust or other obstructions, since its area is a critical parameter which must be maintained accurately constant during the course of the experiment. Because its small size (of the order of a few tenths of a millimeter in diameter) renders it extremely sensitive to degrading influences which may not be evident from visual inspection, the use of a small, constant intensity light source as a reference standard is recommended. By monitoring this reference source periodically with the pinhole both placed and removed, any change in the transmittance of the pinhole will appear as a change in the ratio of these two measurements.

The lenses used in the experiment must be free of aberrations (i.e. diffraction limited), because in the underlying theory outlined in

Reference 1 , the quantity $K_o(r)$ is only independent of the lens aperture when no optical aberrations are present. Otherwise, the uncertainty in $K_o(r)$ (which is integrated with the exponential of the structure function) reduces the accuracy of the evaluation. The magnitude of this error cannot be estimated with any reasonable precision because of the complexity of the calculations involved.

Electronic Component Considerations

In this experiment, the performance required of the electronics presents no special problems. The PM tube output must be amplified and smoothed in such a way that the intensity fluctuations at the center of the focal point are averaged out. Any drift present in the PM tube is of no consequence because only intensity ratios are measured and absolute values are not needed. To minimize the possibility of non-linear effects of the PM tube response influencing the measurement, it is advisable to reduce the light intensity with an attenuating filter during the "wide open" measurement (pinhole removed). If the filter is properly chosen, the tube will be constrained to operate over the same limited portion of its linear range for both measurements. That this procedure does not affect the outcome of the experiment is evident from a study of Equation 7.8. If P is the power measured by the photomultiplier with the filter in place, and T is the transmissivity of the filter, then the true power, which would be measured without the filter (and which is to be compared with P_r) has the value P/T. Upon making this substitution in Equation 7.8, it is seen that T cancels out of the equation as long as the same filter (same T) is used for both the calibration and the actual measurements.

Signal-to-Noise Effects

The theory of the heterodyne equivalent measurement which was described in Reference 1 , does not take into account the effects of noise. Hence, this experiment should be conducted at as high a signal-to-noise level as can be reasonably achieved in order to minimize these effects. After this quasi-noise independent measurement has been made, the effects of noise can be determined, if desired by repeating the measurements at a series of lower power (lower SNR) levels.

ANGULAR FLUCTUATION MEASUREMENT - EXPERIMENT NO. 8

OBJECTIVE

To determine the probability distribution and the frequency spectrum of the angular fluctuations of a laser beam after transmission through the atmosphere.

EXPERIMENTAL PROCEDURE

Description

The experimental arrangement (Figure 10) of Experiment No. 7 (Heterodyne Equivalent Measurement System) can be used for this measurement by replacing the PM tube and pinhole with a moving film and a chopper, respectively. The chopper can be a rotating disc having a ring of uniformly spaced radial slots. When the speed of the film and chopper are properly chosen (see ACCURACY DISCUSSION presented below), the excursions of the focal spot from its center position are recorded on the film as a locus of points whose trace describes the angular fluctuation of the beam as a function of time. To provide a reference for quantitative evaluation of this data, a strobe light is used to mark on the edge of the film the passing of the chopper slot through the average center position of the focal spot.

A variation of this method is to use in place of the film a PM tube whose output is recorded on a moving tape, which is moving with a constant known speed. The electronics accompanying the PM-tube is best designed such that each count produces the same amount of energy. The time intervals of the recorded impulses are a function of focal point displacement in the x-direction. With these data the frequency spectrum of one component of the focal point displacement can be evaluated.

Another similar design is described in Reference 11.

Discussion

Since the phase structure function plays a fundamental role in the theory which underlies both this experiment and the preceding one (No. 7), these two experiments should be performed together to make certain that the same $D_{\phi\phi}$ prevails for both. The simplest way of doing this, provided sufficient power is available in the received beam, is to make use of a beam-splitter which sends one portion of the beam to the pinhole-PM (No. 7), the other to the film (No. 8).

Evaluation of P(y) from the developed film can be accomplished with the aid of an elongated narrow slit which is placed parallel to the film, between it and a PM tube recorder. When the moving film is illuminated from the side opposite the PM, the integrated output of the PM measures the total light flux passed through the developed images lying at the transverse distance y determined by the position of the slit. By repeating this readout for all values of y, the relative probability P(y) is obtained.

By a similar technique, using a transverse (y-direction) slit which extends across the whole width of the film, and recording all PM outputs at the same amplitude, the resulting sequence of pulses can be used to calculate the frequency spectrum of the x-component (along the direction of the film) of the angular fluctuations.

The isotropy of the angular fluctuations can be checked by optically rotating the direction of the measurement by 90 degrees and then repeating the experiment.

SUMMARY OF ERRORS

The following experimental factors directly affect the accuracy of the measurement:

Drive Speed of the Film - must be fast enough to permit recording of high frequency angular fluctuations. The minimum film speed, v is set by the condition

$$v_f > r_i/T$$

where r_i is the radius of the focused image and T is the period corresponding to the highest frequencies being measured.

Chopper Speed - must be great enough so that the exposure frequency, f_e is at least twice as large as the highest frequency being measured. The accuracy in determining the amplitude A_ω of the frequency spectrum for the frequency $f = \omega/2\pi$ is described by the expression

$$A_\omega \pm \sigma\sqrt{\frac{1}{f_e\tau}}$$

where σ is the standard deviation of the image location and τ is the sampling time.

Width (Δy) of Integrating Slit - must not be greater than the uncertainty of the image location in the y-direction.

Fluctuations in Intensity of Transmitted Light - applies to the readout of the film data. The intensity of the light transmitted through the developed film may vary from zero (between images) to several times the average intensity (as a result of scintillations), with a standard deviation equal (approximately) to the average intensity.

Ensemble Size - the number of images N_i included in the integration determines the uncertainty in $P(y)$ through the term

$$\frac{P(y)}{\sqrt{N_i}} = \frac{P(y)}{\sqrt{P(y)(W+d)\ N_e}}$$

where W and d are the widths of the scanning slot and image spot, respectively, and N_e is the total nunber of image spots on the film.

Measurement Discussion

The measureables being sought in this experiment are the probability distribution of the angular fluctuation of the laser beam, and the frequency spectrum of this motion.

For estimating the order of magnitude of the angular fluctuation which may be expected, the observed "dancing" of stellar images provides a useful reference. The size of the stellar image formed by a telescope is known to be larger than the Airy disc diffraction limit. For small aperture telescopes it is found that the lateral excursions of the image are inversely proportional to the diameter of the aperture, down to about 7.5 cm, below which good data is not available. Because of the integrating effect of large apertures, the focused image in such cases assumes the form of an enlarged, diffuse "Tremor Disc." Under average conditions, the angular amplitude of the image excursion is about 10 seconds of arc for stars at the zenith and about 20 seconds for an elevation of 30°.

Assuming the effective focal distance $R_1 = 1.25 \times 10^4$ cm. , which corresponds to a 0.01 cm pinhole, a 0.1 cm spot and a 5 cm aperture, the image point excursion corresponding to 10 arc sec (5×10^{-5} radians) is $(1.25 \times 10^4) \times (5 \times 10^{-5}) \simeq 0.6$ cm. The ratio of the average excursion to the diameter of the image is a significant parameter which determines the accuracy of the angular fluctuation measurement. It can vary in magnitude from fractional values up to the order of several times unity, depending on the experimental conditions.

An estimate of the frequency of the image "dancing" can also be deduced from astronomical data. For typical conditions, the average frequency is around 40 cps, but frequencies up to 1000 cps have been measured.

Data Evaluation

After the recording film has been exposed as described, and then developed, the probability distribution of the angular fluctuation of the laser beam and the frequency spectrum of this motion can be evaluated by methods which will now be described. For convenience in discussing these procedures the film will be assigned the coordinates "x," along the direction of its motion and "y" in the transverse direction.

Angular Fluctuation

The distribution of the y-component of the angular image displacement can be measured by means of an elongated slit positioned parallel to the x-direction at a specified transverse position y. The film is situated between the slot and a PM tube which records the light passing through the film at the position of the slot. When the film is moved across the slot in the x-direction, the light measured is that which is passed by the developed images located at the coordinate y on the film. This measurement is repeated over all y-coordinates and the light flux is integrated over the same film length for each. Since the integrated flux for each measurement is proportional to the relative probability that the image lies at that value of y, a plot of this flux as a function of the deviation of y from its mean value yields the angular distribution of one component of the fluctuation.

Frequency Spectrum

Evaluation of the frequency spectrum can be accomplished by passing the film across a slot which extends over the full width of the film, and measuring the transmitted light as before, with a PM tube placed behind the slot. By clipping the electronic pulses which result from the light transmitted through each passing image, so that the outputs are all recorded at the same amplitude, this sequence of pulses can be used to calculate the frequency spectrum of the angular fluctuations. (Its cosine transform gives the auto-correlation function of the x-component of the angular displacement.)

The two techniques just described provide only one component of each measured quantity: the y-component of the angular displacement and the x-components of the angular velocity and frequency spectrum. If greater complexity in the data processing equipment is acceptable, such as the use of a flying-spot scanner, then both components can be evaluated in the same experiment. However, from the theory there is no reason to expect the statistics to depend on the orientation of the measurement. A check of this assumption is easily made by optically (e.g., using a prism or mirror) or mechanically rotating the direction of the measurement by 90 degrees.

Accuracy Discussion

Several design parameters have a direct bearing on the accuracy of the measurements. An obvious example is the drive speed of the film, which affects the frequency resolution achievable in measuring the frequency spectrum of the angular fluctuations. If the film moves too slowly the fine structure present in the trace of the fluctuations will be lost, especially when rapid fluctuations are coupled with small deflections. Scintillation effects also reduce the accuracy of this measurement. This frequency resolution criterion can be used to establish the minimum permissible film speed. If T is the period of the highest frequencies being measured and r_i is the radius of the focused image, then the minimum film speed v_f is expressed by the condition

$$v_f \quad r_i > /T.$$

Another important consideration is the speed of the chopper slot in relation to the film speed. This must be great enough to ensure that the focused spot remains relatively stationary during the time that the moving slot is exposing it to the film. This requirement is similar to that imposed on a slit-shutter camera when it is used to photograph a rapidly moving object, such as a speeding car. If the shutter speed is too slow, the image of the car will be elongated or foreshortened, depending on the relative directions of motion of the car and the shutter. Similarly, if the chopper is rotated too slowly, the image will be smeared out on the film and its intensity will be correspondingly reduced. Intensity fluctuation will also occur as a result of atmospheric scintillation and must be taken into account in evaluating the film data.

An additional constraint on the chopper speed arises from the requirement that the number of exposures per second, f_e must be at least twice as large as the highest frequency being measured. For a given frequency band, the accuracy in measuring the amplitude A is described by the expression

$$A_f \pm \sigma_x \sqrt{\frac{1}{f_e \tau}} \,,$$

where σ_x is the standard deviation of the image location and the product of f_e and the sampling time τ (which is the time constant of the averaging circuit) is the "ensembel number," N .

The accuracy of the angular displacement measurement is independent of the length (in the x-direction) of the slit used for integrating the transmitted light. Its width, however, should be no larger than the uncertainty of the image location in the y-direction. In evaluating this measurement, the fact that the intensity of the light passing through the film is highly variable must be taken into account. In fact the intensity may range from zero (between images) to several times the average intensity (as a result of scintillations). Roughly speaking, its standard deviation is proportional to the average image brightness.

The result of each measurement at the coordinate y is an amount of integrated light energy L(y), which is the summation of n_v crossings of the trace on the film. The probability density P(y) of the image spots as a function of y is

$$P(y) = \frac{L(y)\,\Delta y}{\int_{\infty}^{\infty} L(y)dy}$$

where Δy is the uncertainty in the position of the image spot.

The accuracy of the measurement depends on the size of the ensemble, i.e., on the total number N_i of images included in the integration. If the widths of the scanning slot (in the y-direction) and the image spot are designated W and d, respectively, and N_e is the total number of spots on the film, then

$$N_i = P(y)\ (W+d)\ N_e.$$

Then, assuming that the standard deviation of the image brightness is equal to its average value, the uncertainty in P(y) is

$$\frac{P(y)}{\sqrt{P(y)\ (W+d)\ Nd}}$$

SPREAD FUNCTION AND BEAM SPREADING MEASUREMENT - EXPERIMENTS 9 AND 10

Because of their common objective and theoretical basis, i.e. measurement of the spreading of a laser beam due to the atmosphere, these two experiments are discussed together.

OBJECTIVES

Experiment No. 9

To measure the spreading of the intensity pattern of a laser beam transmitted through the atmosphere, as a function of the distance and the size of the transmitter aperture.

Experiment No. 10

To measure the spread of a laser beam, its total power and power distribution at a distance, as a function of the transmitter aperture. (Note: This experiment differs from Experiment No. 9 only in its implementation, which enables beamspreading data to be obtained at greater distances and for large beamwidths.)

EXPERIMENTAL PROCEDURES

Experiment No. 9

Description

Using only the transmitter optics (Figure 12), a laser beam is focused onto a recording film (without optics) placed at a known distance from the transmitter. This procedure is repeated for a series of different distances and transmitter apertures, and the size of the focused image, as measured from the developed film, is then recorded as a function of these variables. At distances where the spread of the beam becomes too large for the film,

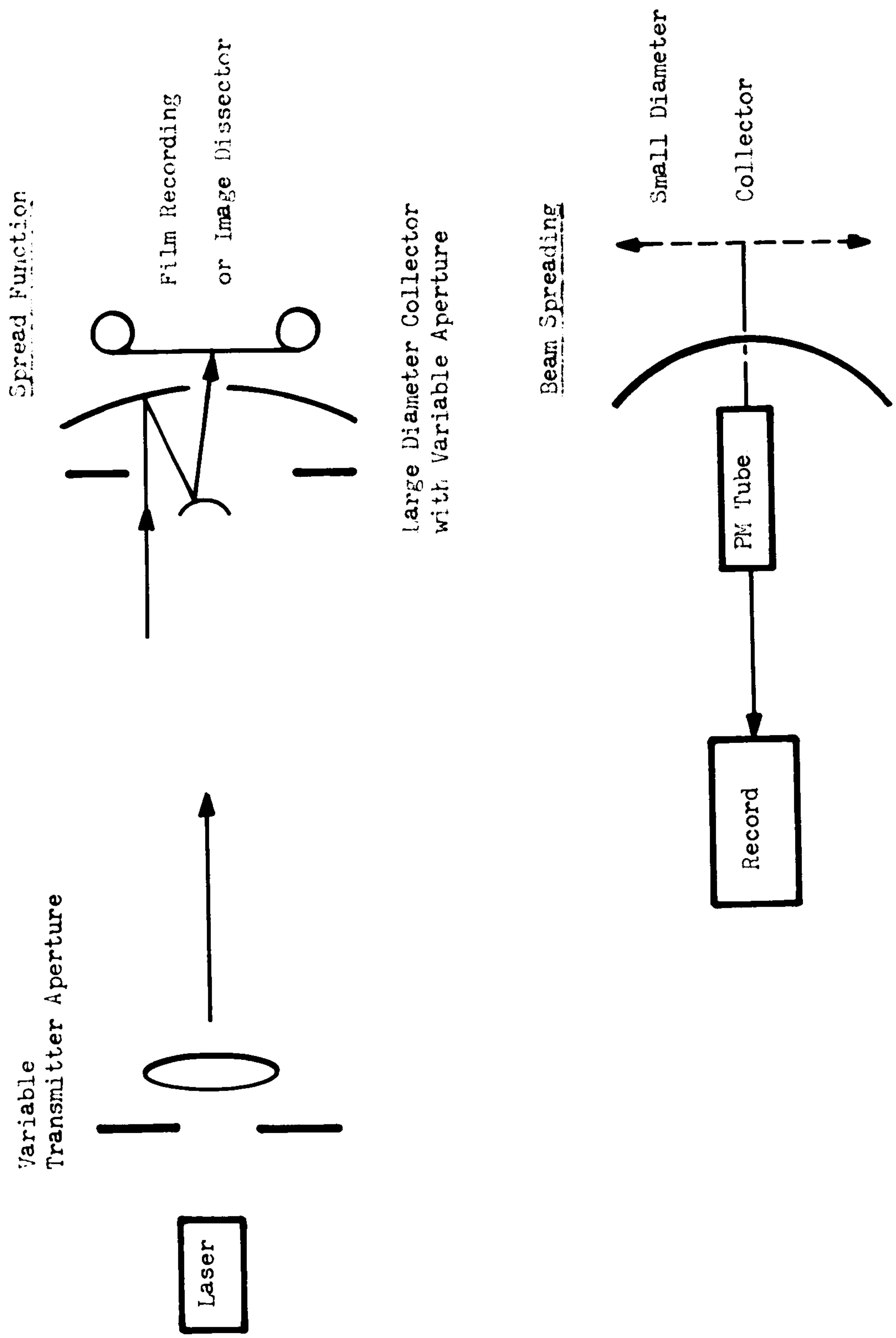

Figure 12. Measurement of Spread Function and Beam Spreading.

a Fresnel lens can be used to reduce it and also to increase the irradiance and thereby decrease the required exposure time.

Discussion

Since it is the spreading due to the atmosphere which is being measured, the transmitter optics must be of high optical quality in order to minimize undesirable distortions in the transmitted beam. The nature and magnitude of the residual optical system aberrations can be determined from the measurements made at very short distances.

Experiment No. 10

Description

A laser transmitter having a variable aperture and focal length is used, as in Experiment No. 9. At a series of far distances (greater than those used in Experiment No. 9), the irradiance of the beam, and when possible, its total power, are measured for different settings of the transmitter aperture. For each measurement the focus of the transmitter is adjusted to an optimum setting at which the size of the beam at the receiver is a minimum.

Two types of receivers are used, depending on the width of the beam. A single, large aperture receiver is used when the aperture is larger than the beamwidth. After the receiver is set coaxially with the beam, the irradiance distribution is measured with a series of reduced apertures and the total power is measured at the maximum aperture.

When the receiver diameter is smaller than the beamwidth, the irradiance distribution is measured with a PM tube which is moved across the beam along two perpendicular lines passing through its center. The total power in this case is obtained by integration over the beam. For this measurement, a standard light source is needed to calibrate the PM tube.

Discussion

The remarks made under Experiment No. 9 are also applicable to this experiment. In fact, the two experiments should be conducted jointly, since they measure the same phenomenon.

ACCURACY DISCUSSION

In these experiments, the inherent variability and uncertainty of the phenomenon being measured (e.g. the intensity distribution within the laser beam) is so much greater in magnitude than the errors associated with the techniques of measuring them, that a formal error analysis has little practical significance. For this reason, the discussion presented here is qualitative in nature and simply describes, in a general way, the conduct of the experiments with regard to some of the practical cautions and considerations which must be heeded if the chances for a successful outcome of these relatively difficult experiments are to be maximized.

The experiment (No. 9) should be carried out beginning at a short distance, e.g. 1 km or less, using several aperture sizes. For each aperture the focus adjustment must be used to minimize the size of the image at the receiver, noting whether such adjustment follows a predictable pattern, i.e. whether the focus condition can be pre-set on the basis of known range and other experimental parameters, or whether in fact the random nature of the atmospheric-optical "system" effectively prevents this condition from being determined in advance.

The size of the image, averaged over a few seconds, must be measured, which is easily done at short range by directly exposing a film and measuring the developed image (allowing for spreading effects in the film). At longer ranges where the image becomes larger, a Fresnel lens can be used to reduce the size of the beam and increase its irradiance sufficiently to permit the film technique to be applied here also.

Still further away (No. 10), measurement of the intensity distribution across the beam will require a moveable PM tube mounted on an optical bench. Preferably, the irradiance should be measured along two perpendicular lines (horizontal and vertical) passing as nearly as possible through the center of the beam. The latter condition, which is necessary for correctly evaluating the intensity distribution of the beam, will require careful alignment of the receiver, assisted by some preliminary measurements to establish the center of the beam. In order to provide a check on the distribution being measured, the total power integrated over an extended (but known) area of the beam should also be measured. These two measurements should be made with the same PM tube.

As long as the beam is narrow enough so that it can be sampled over its whole width, there is no need to calibrate the sensor (film or PM tube) to an absolute standard, since only the relative irradiance distribution is required. (Determination of the power loss due to atmospheric absorption is not an objective of this experiment). When, on the other hand, the beam is so wide that only a portion of its width can be explored, the sensor should be calibrated with a standard light source, and a large aperture measurement should be made in addition to the horizontal and vertical scanning measurements.

This, together with a knowledge (or estimate) of the atmospheric absorption, will allow extrapolation of the data (based on an appropriate model for the intensity distribution) for purposes of drawing some reasonable conclusions about the extent and distribution of the whole beam.

The most reliable measurements may be expected for horizontal links over and close to a level stretch of ground, such as a long, straight, flat road. When the environmental conditions become different as the measurement distance is changed, such as over an elevated path passing over variable terrain, or when an airborne terminal is used, and the complications of tracking errors are added to the problem, it becomes difficult to obtain meaningful results. The optical links should therefore be carefully chosen with an eye to minimizing these complicating factors, at least until the effects of each have been assessed separately.

POLARIZATION FLUCTUATION MEASUREMENT - EXPERIMENT NO. 11

OBJECTIVE

To detect and measure any changes in the state of polarization of a laser beam which may result from transmission through a turbulent atmosphere. (Note: This is a revised experiment, and is similar to the experiment reported in Reference 12.)

EXPERIMENTAL PROCEDURE

Description

A beam of linearly polarized light from a laser is transmitted through a region of turbulent air. At the receiver (Figure 13), the beam is reduced in size, filtered, and divided by a Wollaston prism into two components whose polarization vectors are perpendicular to each other. This results from orienting the prism 45 degrees to the plane of polarization of the beam. Separate PM-tubes measure the intensity of each component, and the output of each, as well as the difference between them, is recorded. Over short transmission paths, such as in a laboratory test, or in a vacuum, the difference in the PM outputs should be zero. After this calibration has been made, any atmospheric polarization effects present would be indicated by a non-zero difference in the outputs.

Since the same null result can be obtained even with an unpolarized wave, the linear polarization of the beam should be tested by rotating the prism until one component of the beam is extinguished, as indicated by a zero output of the corresponding PM. This condition should occur after the prism has been rotated 45 degrees from its initial orientation, when the polarization

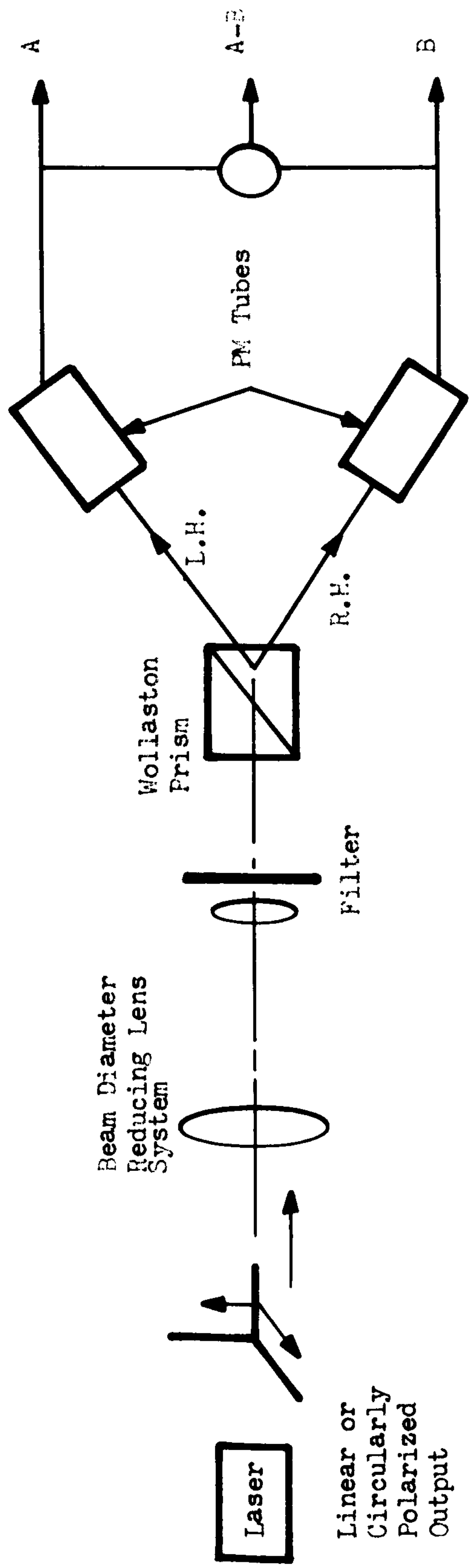

Figure 13. Measurement of Polarization Fluctuation

planes of the beam and prism should be perpendicular.

This measurement should also be made with circularly polarized light.

An alternative technique, (Figure 14), similar to that described in Ref. 1, makes use of a rotating polarization analyzer to extract the plane of polarization. Any deviation from this polarization plane is indicated by a phase shift relative to a rotating reference phase. This system will be described in detail in the forthcoming Task III report.

Remarks

Measurements made with circularly polarized light will show if the atmosphere affects both planes of polarization in the same way or whether, as suggested by the cited experiment of Fried and Mevers (Reference 12), this is not the case.

This experiment should be conducted at a high signal-to-noise ratio.

SUMMARY OF ERRORS

The error terms appearing in the evaluation of $\langle \sin 2x \rangle_{rms}$ (x = angle of rotation of polarization plane) are

Noise:

$$\frac{1}{\eta \bar{n}_\tau} \frac{1}{\sqrt{n-2}}$$

where η = quantum efficiency of the PM

$\bar{n}_\tau$ = average number of photons arriving during time τ the time constant of the receiver circuit

n = number of repeated measurements

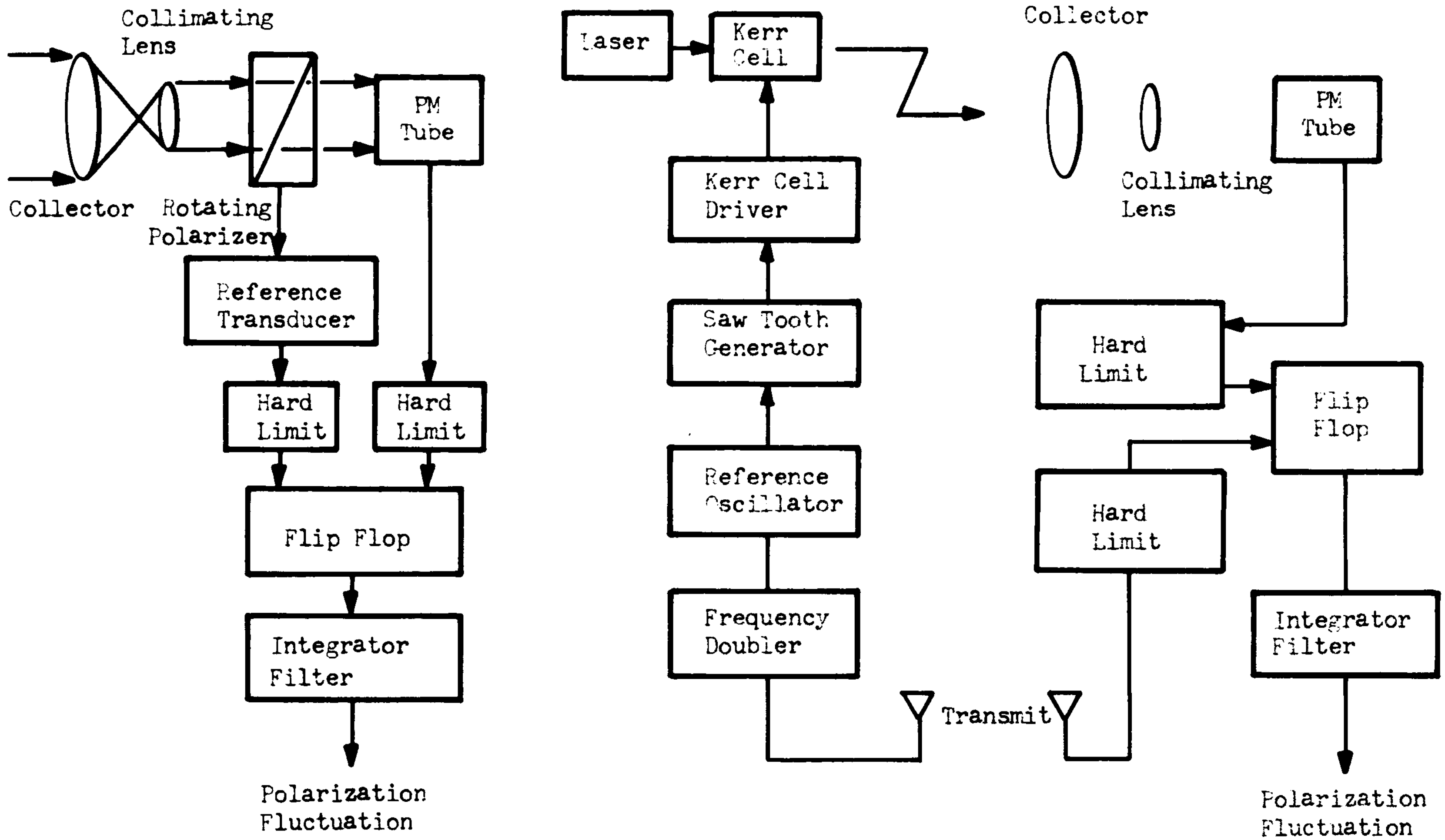

Figure 14. Alternate Configuration for Polarization Fluctuation Measurement.

Standard Deviation: $\left(\left\langle X^2 - \langle X^2\rangle\right\rangle\right)^{\frac{1}{2}} \left(1 \pm \frac{1}{2}\sqrt{\frac{2}{n-2}}\right)$

where X is a measure of the polarization as defined later

The above terms combine as follows:

$$\langle \sin^2 2x\rangle = \langle (X^2 - \langle X\rangle^2)\rangle \pm \sqrt{\langle (X^2 - \langle X\rangle^2)^2\rangle \frac{2}{n-2} + \left(\frac{1}{\eta \bar{n}_T}\right) \frac{1}{n-2}}$$

ERROR ANALYSIS

After reflection from the Wollaston prism, the two perpendicularly polarized components of the beam are described by the expressions

$$A \sin (45° + x) \text{ and } A \cos (45° + x)$$

where A is the amplitude of each and x, the angular rotation of the polarization plane is to be measured. Allowing for a difference in gain between the two PM tubes, which is expressed by the two different amplitudes A_1 and A_2, the PM outputs are

$$\frac{A_1^2}{2} (\cos x \pm \sin x)^2 \quad \text{and} \quad \frac{A_2^2}{2} (\cos x - \sin x)^2$$

or

$$\frac{A_1^2}{2} (1 + \sin 2x) \quad \text{and} \quad \frac{A_2^2}{2} (1 - \sin 2x) \qquad (11.1)$$

Letting $A_2^2 = A_1^2 (1 - \epsilon)$, the difference Δ in the two outputs can be written

$$\Delta = \frac{A_1^2}{2} (1 + \sin 2x) - \frac{A_2^2}{2} (1 - \sin 2x)$$

$$= \frac{A_1^2}{2} \left[1 + \sin 2x - 1 + \sin 2x + \epsilon (1-\sin 2x)\right]$$

$$= A_1^2 \left[\sin 2x + \frac{\epsilon}{2} (1 - \sin 2x)\right] \qquad (11.2)$$

Similarly, the sum Σ of the output is

$$\Sigma = \frac{A_1^2}{2}(1 + \sin 2x) + \frac{A_2^2}{2}(1 - \sin 2x)$$

$$= \frac{A_1^2}{2}\left[1 + \sin 2x + 1 - \sin 2x - \epsilon(1 - \sin 2x)\right]$$

$$= A_1^2\left[1 - \frac{\epsilon}{2}(1 - \sin 2x)\right] \qquad (11.3)$$

Forming Δ/Σ gives

$$\frac{\Delta}{\Sigma} = \frac{\sin 2x + \frac{\epsilon}{2}(1 - \sin 2x)}{1 - \frac{\epsilon}{2}(1 - \sin 2x)}$$

$$\simeq \sin 2x + \frac{\epsilon}{2}(1 - \sin 2x) \qquad (11.4)$$

Since the polarization effect being measured is expected to be quite small, if it can even be detected at all, the assumption $\sin 2x \ll 1$ is valid and Equation 11.4 can be simplified:

$$\frac{\Delta}{\Sigma} \equiv X = \sin 2x + \frac{\epsilon}{2} \qquad (11.5)$$

From Equation 11.5 it is evident that the accuracy of the polarization measurement is extremely sensitive to the size of the error term, $\epsilon/2$. An $\epsilon/2$ value of only 1 percent corresponds to an angular error of 0.6°, which must be considered large for this type of measurement. Hence, the difference in gain between the two PM tubes (which determines ϵ) must be held to a very low value.

When the prism is rotated by 45 degrees to test the polarization of the beam, X becomes, (letting $X = 45° + x'$)

$$X_2 = \frac{\sin 2(45° + x') + \frac{\epsilon}{2}\left[1 - \sin 2(45° + x')\right]}{1 - \frac{\epsilon}{2}\left[1 - \sin 2(45° + x')\right]}$$

$$= \frac{\cos 2x' + \frac{\epsilon}{2}(1 - \cos 2x')}{1 - \frac{\epsilon}{2}(1 - \cos 2x')}$$

When x' is small $x' \ll 1$, $\cos 2x' \simeq 1 - 2x'^2$ and X_2 becomes

$$X_2 = 1 - 2x'^2 + 2\epsilon x'^2$$

$$\cong 1 - 2x'^2$$

which reduces to unity when $x' = 0$. With this condition, the output of one PM tube should be zero, since, with $x = 45°$, $\sin 2x = 1$ in Equation 11.1. This may require calibration of the PM tubes.

Standard Deviation of X

From the definition of X (Equation 11.5),

$$\left\langle X \right\rangle = \frac{\epsilon}{2} \pm (\sin 2x)_{rms} \sqrt{\frac{1}{n-2}}$$

$$(\sin 2x)_{rms} = \left(\left\langle X^2 - \left\langle X \right\rangle^2 \right\rangle\right)^{\frac{1}{2}} \left(1 \pm \sqrt{\frac{2}{n-2}}\right)^{\frac{1}{2}}$$

$$= \left(\left\langle (X^2 - \left\langle X \right\rangle^2 \right\rangle\right)^{\frac{1}{2}} \left(1 \pm \frac{1}{2}\sqrt{\frac{2}{n-2}}\right)$$

where $\sin 2x \simeq 2x$ and n is the number of measurements.

Noise

The noise current Δi_n which competes with the signal is the sum of the noise power from the outputs of the two photomultiplier tubes. The noise power from each PM tube has a rms value $\dfrac{A^2}{2\sqrt{\eta \bar{n}_\tau}}$ where η is the quantum efficiency

and $\bar{n}_\tau$ is the average number of photons which arrive in time τ; τ is approximately the time constant of the circuitry following the photodetectors. When the noise term is added to the error in amplitude, X becomes,

$$X = \frac{2 \sin 2x' + \epsilon(1 - \sin 2x') + \dfrac{\Delta i_n}{A_2}}{2 - \epsilon(1 - \sin 2x') + \dfrac{\Delta i_n}{A_2}}$$

This gives approximately

$$X \cong \sin 2x' + \frac{\epsilon}{2} + \frac{\Delta i_n}{A_2}$$

Thus

$$\left\langle (\sin 2x')^2 \right\rangle = \left\langle (X^2 - \left\langle X \right\rangle^2)^2 \right\rangle \pm \sqrt{\frac{2\left\langle (X^2) - \left\langle X \right\rangle^2 \right\rangle^2}{n-2} + \frac{1}{(\eta \bar{n}_\tau)^2 (n-2)}}$$

when the experimental data consists of n measurements.

ATMOSPHERIC BACKSCATTER, TRANSMITTANCE, AND SKY RADIANCE - EXPERIMENT NO. 12

This is a revised experiment which is presented in place of Experiments 12, 13, 14, 15, 16 and 17 as proposed in Volume 1 of the Task II report.

OBJECTIVE

To measure atmospheric backscatter, transmittance, and sky radiance.

SIGNIFICANCE

Laser light scattered back to the receiver may interfere with reception. Data is needed to indicate the magnitude of the interference and the circumstances under which it may occur.

Atmospheric transmittance influences power required, favoring some wavelengths over others. Data would verify the applicability of previous studies, done over relatively broad spectral bands, to the use of monochromatic laser light.

The radiance of the sky provides a limiting background for signals from space. Sky luminance data are inadequate to assess this limitation accurately because they describe the visual effect of the optical spectrum. Needed are spectral radiance measurements at particular laser wavelengths.

EXPERIMENTAL PROCEDURE

Description

The principal elements of the receiver are a Cassegrain telescope, an optical filter, and a photomultiplier, as shown in Figure 15. The laser beam is transmitted along the axis of the telescope by reflecting it with a plane mirror located in front of the secondary mirror. A power monitor

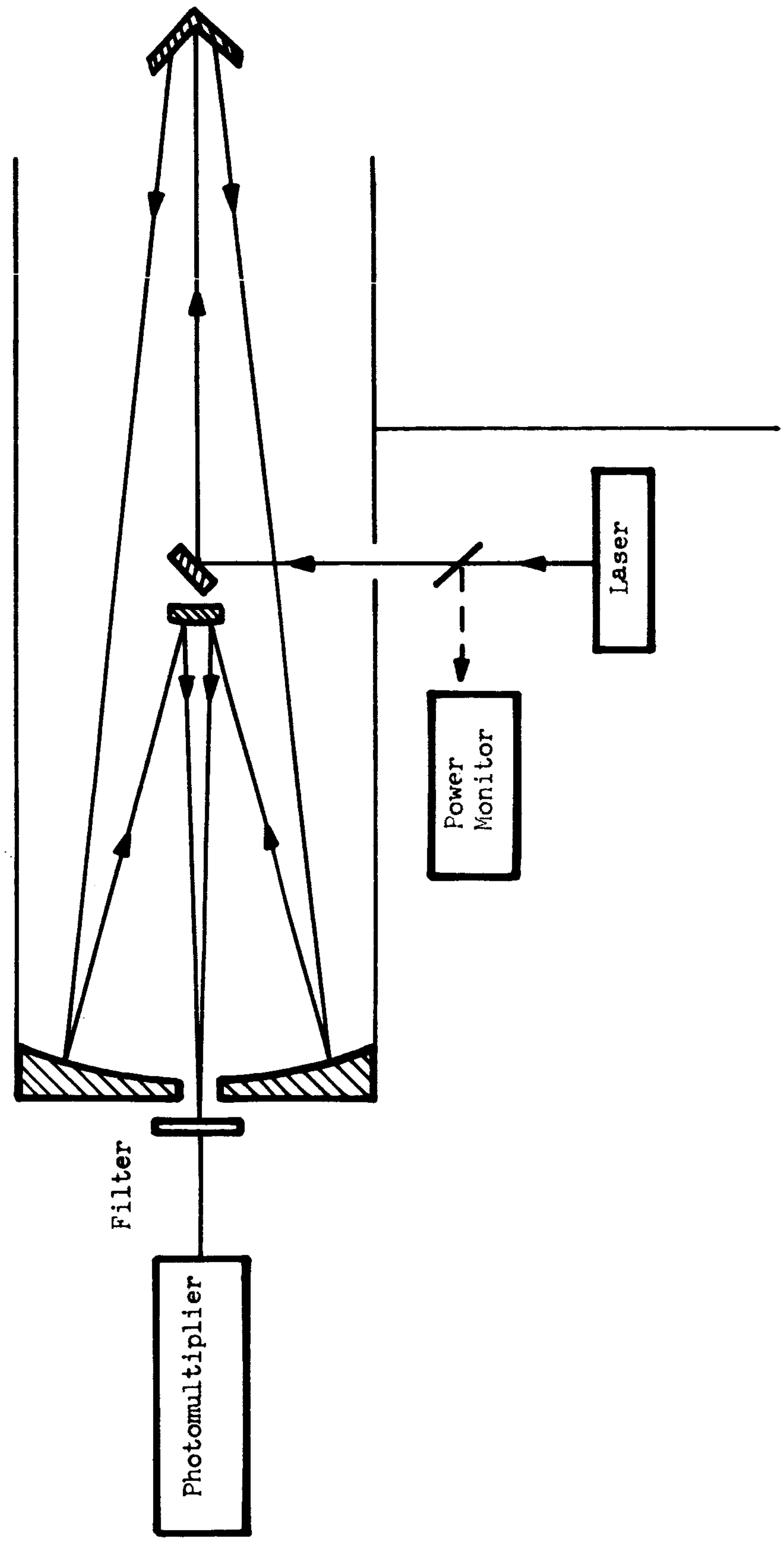

Figure 15. Measurement of Atmospheric Backscatter, Transmittance, and Sky Radiance.

measures the small fraction of transmitted power reflected to it by a beam splitter. A retro-reflector is placed a great distance in front of the telescope. The laser beam is turned on and off. The light producing the resulting photomultiplier current consists of sky background within the pass-band of the optical filter, laser light backscattered from the air, and laser light reflected back by the reflector. The photomultiplier current at different times indicates sky radiance, backscatter from different volumes along the light path, and transmittance of the air column extending from the telescope to the retro-reflector.

SUMMARY OF ERRORS

Sources of Error

Measurement limitations, calibration drift and instability, and movements of the retro-reflector constitute the principal sources of error. For example, the photomultiplier gain may drift or fluctuate. Vehicle - or wind-induced movements of the retro-reflector introduce fluctuations of A_m and Ω_m.

Also, the sky background may obscure some of the measurements of backscatter or transmittance. However, the error analysis neglects this veiling effect, but rather it deals with random errors of the measurements.

Expressions for Errors

A mean-square fractional error in x is indicated as $E(x)$. The fractional errors are assumed to be small compared to unity, random and statistically independent for the different measured quantities. Then the derived quantities are as follows.

Atmospheric backscatter is given by

$$(kP_tA_r)^{-1}\ I_{sc}\ \left\{1 \pm \left[E(I_{sc}) + E(P_t) + E(A_r) + E\ (k)\right]^{\frac{1}{2}}\right\}$$

where I_{sc} is photocurrent produced by the backscatter, k is a constant which includes the transmittance of the optics and the responsivity and gain of the photomultiplier, A_r is the aperture area of the telescope, and P_t is the radiant power sent out by the transmitter.

Atmospheric transmittance is given by

$$R_m{}^2\left[(I_m\ \Omega_m\ \Omega_t)/(k\ A_rA_mA_t)\right]^{\frac{1}{2}} \quad x$$

$$\left\{1 \pm \left[16\ E(R_m) + E(I_m) + E(\Omega_m) + E(\Omega_t) + E(k) + E(\rho) + E(A_r) + E(A_m) + E(P_t)\right]^{\frac{1}{2}}\right\},$$

where R_m is distance from the telescope to the retro-reflector, I_m is the photocurrent produced by the light reflected back from the reflector, Ω_m is the solid angle into which the reflector reflects the intercepted power, Ω_t is the solid angle of the emitted laser beam, ρ is the radiant reflectance of the reflector, and A_m is its cross-sectional area.

Sky radiance is given by

$$(k\ A_r\,\Omega_r)^{-1}\ I_b\ \left\{1 \pm \left[E(I_b) + E(k) + E(A_r) + E(\Omega_r)\right]^{\frac{1}{2}}\right\},$$

where I_b is the photocurrent produced by sky background and Ω_r is the solid angular field of view of the telescope.

MEASUREMENT ANALYSIS

The current I_b produced by sky background is related to the sky radiance N_{sk} as

$$I_b = k\, N_{sk}\, A_r\, \Omega_r \quad , \tag{12-1}$$

where k is a constant which includes the transmittance of the optics and the responsivity and gain of the photomultiplier, A_r is the aperture area of the telescope, and Ω_r is the solid angular field of view.

The transmittance τ of the air column extending a distance R from the telescope depends upon the extinction coefficient γ of the air through the relation

$$\tau = \exp\left(- \int_0^R \gamma(x)\, dx \right) \quad , \tag{12-2}$$

where x is distance along the light path. Assuming that the field of view of the receiver exceeds the width of the transmitted beam, we can show that the current I_{sc} produced by scattered light at time t after the beam is turned off is given by

$$I_{sc} = k\, P_t\, A_r \int_{R_1}^{R_2} b\, \tau^2\, dR/R^2 \quad , \tag{12-3}$$

where

$$R_1 = \tfrac{1}{2}\, c\, t,\ R_2 = \tfrac{1}{2}\, c(t + t_o) \ ,$$

c being the speed of light and t_o the duration of the on-time. The quantity b equals 4π times the optical radar cross-section per unit volume of air, and P_t is the power emitted by the laser transmitter.

The current I_m produced by the light reflected back by the retro-reflector at a distance R_m is found to be

$$I_m = \frac{k \rho \tau^2 A_r A_m P_t}{R_m^4 \Omega_m \Omega_t} \qquad (12\text{-}4)$$

where ρ is the reflectance of the retro-reflector, τ is the transmittance of the path from 0 to R_m , A_m is cross-sectional area of the retro-reflector, Ω_m is the solid angle into which it reflects the intercepted power, and Ω_t is the solid angle of the transmitted laser beam.

The background current I_b is as if the receiver faced a surface of radiance N_{sk} without intervening air. Suppose that this surface were at a distance R. The receiver sees an area $\Omega_r R^2$, and it subtends a solid angle A_r/R^2 . Because radiance is radiant power emitted into unit solid angle by a unit area of surface, the relation (12-1) follows.

The receiver must be calibrated in terms of the factors k, A_r, and Ω_r . Then measurements of I_b yield values of N_{sk} with the aid of Equation (12-1). At the same time as I_b is measured, the azimuth and zenith angles of the sun and telescope and the appearance of the sky should be recorded because sky radiance depends upon atmospheric turbidity and the relative position of the sun.

Now suppose that the laser is turned on at time $t = - t_o$ and then off at time zero. For time greater than zero, the telescope receives radiant energy scattered back from the air and reflected by the retro-reflector. The reflected power returns between $t = 2(R_m/c) - t_o$ and $t = 2 R_m/c$.

The backscattered power received at time t comes from a volume lying between $R_1 = \frac{1}{2}$ c t and $R_2 = \frac{1}{2}$ c(t + t_o) , which comes from integrating the returns from elementary volumes lying between R_1 and R_2. The irradiance H_L (power per unit area) of the laser beam at distance R is given by

$$H_L = \tau P_t/\Omega_t R^2 \quad . \tag{12-5}$$

The power dP_{sc} received by the telescope from the elemental volume dV = $\Omega_t R^2 dR$ is given by

$$dP_{sc} = \tau b \; H_L \; dV(A_r/R^2) \quad , \tag{12-6}$$

or in view of Equation (12-5), it reduces to the form

$$dP_{sc} = b \tau^2 P_t A_r dR/R^2 \quad . \tag{12-7}$$

In general, b and τ are functions of R so that they must be included under the integral sign when integrating to obtain Equation (12-3).

We assume that the laser beam fully illuminates the retro-reflector, which accordingly intercepts radiant power of the amount τP_t $(A_m/R_m{}^2 \Omega_t)$ Then we further assume that the reflected laser light fully illuminates the receiver telescope, the fraction ρ of the intercepted power being returned within a solid angle Ω_m which is greater than $A_r/R_m{}^2$. These assumptions lead to the relation (12-4).

From measurements of I_m and R_m, the transmittance τ can be obtained as a function of R_m, path location and orientation, and atmospheric conditions. Backscatter, however, must be recorded in the form $I_{sc}/(kP_tA_r)$ as a function of time and pulse length, which can be interpreted in terms of the distance and volume of the backscattering region.

ERROR ANALYSIS

A measure M of the backscatter is the ratio

$$M = (k\ P_t\ A_r)^{-1}\ I_{sc} \quad .$$

Assuming that the fractional errors of measuring these quantities are small compared to unity, we can relate them in terms of the differentials of the logarithms of both sides, obtaining the result that

$$dM/M = d\ I_{sc}/I_{sc} - dk/k - dP_t/P_t - dA_r/A_r \quad .$$

Assuming further that the errors are random and that any one variable is statistically independent of the others, we can equate the mean square of the resultant error to the sum of the mean squares. Letting σ denote root-mean-square error and E fractional mean-square error, we obtain the formula

$$\sigma_M = M \left[E(I_{sc}) + E(k) + E(P_t) + E(A_r)\right]^{\frac{1}{2}} \quad .$$

Consequently, atmospheric backscatter is given by

$$M \pm \sigma_M = (k\ P_t A_r)^{-1}\ I_{sc} \left\{1 \pm \left[E(I_{sc}) + E(k) + E(P_t) + E(A_r)\right]^{\frac{1}{2}}\right\}.$$

The analysis for transmittance and sky radiance follows along the same lines.

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APPENDICES

The appendices to this report relate to the analysis of errors for experiments Nos. 1 and 3. Appendix A deals with an analysis of beam splitter inaccuracies. Errors introduced by finite data are discussed in Appendix B. A means of defining the variance of the products of log amplitude measurements is given in Appendix C.

APPENDIX A. ERRORS IN PHASE STRUCTURE FUNCTION AND LOG AMPLITUDE CORRELATION FUNCTION CAUSED BY INACCURACIES IN THE BEAM SPLITTER

INFLUENCE OF BEAM SPLITTER INACCURACIES ON PHASE STRUCTURE FUNCTION

In deriving the basic equation (1.1) for the C_i's, it was assumed that the beam splitter induces a $\pi/2$ phase shift to the transmitted beam and that the intensities of the reflected and transmitted light are equal. We assume now that the beam splitter shifts the phase by $\pi/2 + \beta$ and that the ratio of the intensities of the reflected beam to the transmitted beam is $(1+\alpha)/(1-\alpha)$. The quantities C_i are then:

$$\begin{aligned}
C_1(\alpha,\beta) &= \tfrac{1}{2} A_1^2(1-\alpha) + \tfrac{1}{2} A_2^2(1+\alpha) + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}} \cos(\phi_{12}-\beta) \\
C_2(\alpha,\beta) &= \tfrac{1}{2} A_1^2(1-\alpha) + \tfrac{1}{2} A_2^2(1+\alpha) + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}} \sin(\phi_{12}-\beta) \\
C_3(\alpha,\beta) &= \tfrac{1}{2} A_1^2(1+\alpha) + \tfrac{1}{2} A_2^2(1-\alpha) - A_1A_2\,(1-\alpha^2)^{\frac{1}{2}} \cos(\phi_{12}+\beta) \\
C_4(\alpha,\beta) &= \tfrac{1}{2} A_1^2(1+\alpha) + \tfrac{1}{2} A_2^2(1-\alpha) - A_1A_2\,(1-\alpha^2)^{\frac{1}{2}} \sin(\phi_{12}+\beta)
\end{aligned} \qquad (A1.1)$$

We derive now

$$\tan_{(\alpha,\beta)} \phi_{12} = \frac{C_2-C_4}{C_1-C_3} \quad \frac{\alpha(A_2^2-A_1^2) + A_1A_2(1-\alpha^2)^{\frac{1}{2}}\left(\sin(\phi_{12}-\beta) + \sin(\phi_{12}+\beta)\right)}{\alpha(A_2^2-A_1^2) + A_1A_2(1-\alpha^2)^{\frac{1}{2}}\left(\cos(\phi_{12}-\beta) + \cos(\phi_{12}+\beta)\right)}$$

$$\tan_{(\alpha,\beta)} \phi_{12} = \frac{\alpha(A_2^2-A_1^2) + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}}\, 2\sin\phi_{12}\cos\beta}{\alpha(A_2^2-A_1^2) + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}}\, 2\cos\phi_{12}\cos\beta} \qquad (A1.2)$$

This equation shows that with equal intensities of reflected and transmitted light, i.e. with $\alpha=0$, $\tan\ \phi_{12}$ is independent of β. The phase shift or $\pi/2 + \beta$ is allowed to assume any value. The error in the tangent

$$\tan_{(\alpha,\beta)} \phi_{12} - \tan \phi_{12} = \Delta_{(\alpha,\beta)} (\tan \phi_{12})$$

is given by

$$\left(\frac{C_2 - C_4}{C_1 - C_3}\right)_{(\alpha,\beta)} - \left(\frac{C_2 - C_4}{C_1 - C_3}\right) = \frac{\alpha(A_2^2 - A_1^2) + 2 A_1 A_2 (1-\alpha^2)^{\frac{1}{2}} \cos\beta \sin \phi_{12}}{\alpha(A_2^2 - A_1^2) + 2 A_1 A_2 (1-\alpha^2)^{\frac{1}{2}} \cos\beta \cos \phi_{12}} - \frac{\sin \phi_{12}}{\cos \phi_{12}}$$

This reduces to

$$\Delta_{(\alpha,\beta)} (\tan \phi_{12}) = \frac{\alpha(A_2^2 - A_1^2)(\cos \phi_{12} - \sin \phi_{12})}{\left(\alpha(A_2^2 - A_1^2) + 2 A_1 A_2 (1 - \alpha^2)^{\frac{1}{2}} \cos\beta\right) \cos \phi_{12}}$$

Remembering that the error in ϕ_{12} is given by

$$\Delta_{(\alpha,\beta)} \phi_{12} = \Delta_{(\alpha,\beta)} (\tan \phi_{12}) \cos^2 \phi_{12}$$ substitution gives

$$\Delta_{(\alpha,\beta)} \phi_{12} \approx \frac{\alpha(A_2^2 - A_1^2)}{2 A_1 A_2 \cos\beta} (\cos \phi_{12} - \sin \phi_{12}) \qquad \text{(A1.3)}$$

when $\cos\beta \approx 1$ i.e. small errors in beam splitter phase. Evaluating the average and variance of this term gives $\left\langle \Delta_{(\alpha,\beta)} \phi_{12} \right\rangle = 0$ and

$$\sigma_b^2 = \frac{\alpha^2}{2 \cos^2\beta} \left(C_{LL}(0) - C_{LL}(\bar{\rho})\right)$$

In the event that one of the amplitudes (A_1 or A_2) is near zero, the values of the $\{C_i\}$ will not only be equal but also incalculable. This should be born in mind when selecting a particular data reduction scheme. This case should not be included in the calculation of the phase structure function.

INFLUENCE OF BEAM SPLITTER INACCURACIES ON THE CALCULATION OF LOG AMPLITUDE CORRELATION FUNCTION

Using equations 1.3, 1.4, 1.5 and A1.1 expressions for $C_{(\alpha,\beta)}$, $B_{(\alpha,\beta)}$ and finally $A_{1,2(\alpha,\beta)}$ can be written. They are

$$2\,C_{(\alpha,\beta)} = 2\,(A_1^{\,2} + A_2^{\,2}) + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}}\Big[\cos(\phi_{12}-\beta) - \cos(\phi_{12}+\beta) + \sin(\phi_{12}-\beta) - \sin(\phi_{12}+\beta)\Big] \tag{A1.4}$$

$$C_{(\alpha,\beta)} = A_1^{\,2} + A_2^{\,2} + A_1A_2(1-\alpha^2)^{\frac{1}{2}}\sin\beta\,(\sin\phi_{12} - \cos\phi_{12})$$

$$B_{(\alpha,\beta)} = \tfrac{1}{4}\Big\{\Big[\alpha A_2^{\,2} - \alpha A_1^{\,2} + A_1A_2\,(1-\alpha^2)^{\frac{1}{2}}\Big(\cos(\phi_{12}-\beta) + \cos(\phi_{12}+\beta)\Big)\Big]^2 + \Big[\alpha A_2^{\,2} - \alpha A_1^{\,2} + A_1A_2(1-\alpha^2)^{\frac{1}{2}}\Big(\sin(\phi_{12}-\beta) + \sin(\phi_{12}+\beta)\Big)\Big]^2\Big\}$$

$A_{1,2(\alpha,\beta)}$ can now be found; it is

$$A_{1(\alpha,\beta)} = A_{2(\alpha,\beta)} = \frac{C(\alpha,\beta)}{2} \pm \left[\frac{C(\alpha,\beta)^2}{4} \quad B(\alpha,\beta)\right]^{\frac{1}{2}}$$

Using equation A1.4,

$$2\,A_{1,2(\alpha,\beta)} = 2A_1^{\,2} + A_1A_2\sin\beta\,(\sin\phi_{12} - \cos\phi_{12}) + \frac{A_1^{\,2} + A_2^{\,2}}{A_1^{\,2} - A_2^{\,2}}\,A_1A_2\,\sin\beta\,(\sin\phi_{12} - \cos\phi_{12}) - 2A_1A_2\alpha\,(\cos\phi_{12} + \sin\phi_{12})$$

As a means of evaluating $A_{1(\alpha,\beta)}$, the approximation

$$\left(A_{1\,(\alpha,\beta)}^{\,2} - A_1^{\,2}\right) \simeq 2A_1\,\Delta A_1$$

is made; a similar expression holds for $\Delta A_{2(\alpha,\beta)}$.

The result is

$$A_{1(\alpha,\beta)} = \frac{A_1^{\,2}\,A_2}{2(A_1^{\,2} - A_2^{\,2})}\sin\beta\,(\sin\phi_{12} - \cos\phi_{12}) - \frac{A_2}{2}\alpha(\cos\phi_{12} + \sin\phi_{12}) \tag{A1.5}$$

$$A_{2(\alpha,\beta)} = \frac{A_2^2 A_1}{2(A_2^2 - A_1^2)} \sin\beta \; (\sin\phi_{12} - \cos\phi_{12}) - \frac{A_1\alpha}{2} (\cos\phi_{12} + \sin\phi_{12})$$

The error term ΔL_b will be

$$\Delta L_b = \frac{\Delta A_{1(\alpha,\beta)}}{A_1} \log\frac{A_2}{A_o} + \frac{A_{2(\alpha,\beta)}}{A_1} \log\frac{A_1}{A_o}$$

Evaluating this

$$\Delta L_b = \frac{A_1 A_2}{2(A_1^2 - A_2^2)} \left(\log\frac{A_2}{A_o} - \log\frac{A_1}{A_o}\right) \sin\beta \; (\sin\phi_{12} - \cos\phi_{12})$$

$$+ \frac{\alpha}{2}\left(\frac{A_1}{A_2}\log\frac{A_1}{A_o} - \frac{A_2}{A_1}\log\frac{A_2}{A_o}\right)(\cos\phi_{12} + \sin\phi_{12})$$

Then

$$\left\langle \Delta L_b \right\rangle = -\frac{\sin\beta}{4}$$

and the variance of the second beam is

$$\sigma^2_{L_b} = \alpha^2 \frac{(C_{LL}(0) - C_{LL}(\bar{\rho})}{2}$$

APPENDIX B. ERRORS INTRODUCED BY FINITE DATA

The number of data or samples of a random variable (r. v.) limits the accuracy to which the statistical moments of the r. v. may be computed (Reference 3). As an illustration of this fact, the error in measuring the phase structure function will be computed. The error in measuring the log amplitude correlation function can be approached in nearly the same way.

Let $\phi_1 - \phi_2 = \phi_{12}$ and let $E(\phi_{12}) = m$ and $E\left[\left(\phi_{12} - E(\phi_{12})\right)^2\right] = D_{\phi\phi}(\bar{\rho})$. Suppose that there are n sets of data available to calculate the moments. The measured process is

$$E(\phi_{12}^M) = \frac{1}{n} \sum_{i=1}^{n} \phi_{12_i} = m_{12} \quad ; \; m_{12} \text{ is a}$$

random variable since the choice of a limited number of samples cannot reflect all the values of the random variable ϕ_{12}.

The standard deviation of this measurement is

$$\sigma_{12}^2 = E\left[\left(\phi_{12}^M - E(\phi_{12}^M)\right)^2\right]$$

treating each of the variables ϕ_{12}^M; as in sampling theory (i.e. each has the same moment as the random variable they are sampled from) and taking the expectation, there results

$$\sigma_{\phi_{12}}^2 = D_{\phi\phi}^M(\bar{\rho}) - \left(E(\phi_{12}^M)\right)^2$$

This too is a random quantity and in fact is what is measured in the experiment. However, what is wanted is the best estimate of the quantity

$D_{\phi\phi}(\bar{\rho})$ assuming a knowledge of the probability density function of ϕ_{12}. According to Reference 3 , a good estimate of the variance of a random variable x , σ_x^2 , when n data are available is

$$\sigma_x^2 = \sigma_{x_E}^2 \pm \sqrt{E\left\{\left[\sigma_x^2 - \sigma_{x_E}^2\right]^2\right\}}$$

where $\sigma_{x_E}^2$ is the measured variance of the random process x from which the samples were taken. If, as in previous theoretical work (reference 3), it is assumed that ϕ_{12} is a gaussian random variable, then

$$D_{\phi\phi}(\rho) = D^{M}_{\phi\phi}(\rho) \left(1 \pm \sqrt{\frac{2}{n-2}}\right)$$

In the case where ϕ_{12} contains errors due to other random processes such as those connected with the measuring equipment, this expression will be modified and will give the results listed in the summary of errors section.

APPENDIX C. DEFINING VARIANCE OF LOG AMPLITUDE MEASUREMENTS

The expression

$$\left\langle \left(\log \frac{A_1}{A_o} \log \frac{A_2}{A_o} - \left\langle \log \frac{A_1}{A_o} \log \frac{A_2}{A_o} \right\rangle \right)^2 \right\rangle = \Sigma^2 \qquad \text{(B.1)}$$

is to be calculated.

Defining

$$\log \frac{A_1}{A_o} = x_1 = x_{1c} + x_{1r}$$

and

$$\log \frac{A_2}{A_o} = x_2 = x_{2c} + x_{2r}$$

where the subscript indicates those parts which correlate (c) and do not correlate (r) with each other, Equation B.1 becomes

$$\Sigma^2 = \left\langle \left(x_1 x_2 - \langle x_1 x_2 \rangle \right)^2 \right\rangle$$

$$= \left\langle \left(x_{1c} x_{2c} + x_{1c}\, x_{2r} + x_{2c} x_{1r} + x_{1r}\, x_{2r} - C_{LL}(\rho) \right)^2 \right\rangle$$

$$= \Big\langle x_{1c}^2 x_{2c}^2 + x_{1c}^2\, x_{2r}^2 + x_{1r}^2 x_{2r}^2$$

$$+ 2\,(x_{1c}^2 x_{2c} x_{2r} + x_{1c}\, x_{2c}^2 x_{1r} + x_{1c}\, x_{2c}\, x_{1r}\, x_{2r})$$

$$+ 2\,(x_{1c}\, x_{2c} x_{1r}\, x_{2r} + x_{1c}\, x_{2r}^2 x_{1r} + x_{2c} x_{1r}^2 x_{2r})$$

$$- 2\, C_{LL}(\rho)\,(x_{1c}\, x_{2c} + x_{1c} x_{2r} + x_{2c} x_{1r} + x_{1r}\, x_{2r}) + C_{LL}^2(\rho) \Big\rangle$$

With

$$\langle x_1^2 \rangle = \langle x_{1c}^2 + 2\, x_{1c}\, x_{1r} + x_{1r}^2 \rangle$$

$$= C_{LL}(\rho) + \left(C_{LL}(0) - C_{LL}(\rho) \right)$$

$$= C_{LL}(0)$$

we get

$$\Sigma^2 = 3\, C_{LL}^2(\rho) + 2\, C_{LL}(\rho)\left(C_{LL}(0) - C_{LL}(\rho)\right) + \left(C_{LL}(0) - C_{LL}(\rho)\right)^2$$

$$- 2\, C_{LL}(\rho)\left(C_{LL}(\rho)\right) + C_{LL}^2(\rho)$$

$$= C_{LL}^2(0) + C_{LL}^2(\rho)$$